Newsvendor Model for Multi-Inputs and -Outputs with Random Yield: Applications to Agricultural Processing Industries

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Keywords: Agriculture Supply Chain, Applied Operations Research, Stochastic Model Applications, Newsvendor Models.

Abstract: Consider a newsvendor model, which we extend to include both multiple inputs and outputs. Different input types possess different levels of quality, and are purchased at different prices by a processing firm. Each type of input is processed into multiple outputs, which are sold at different prices. The yield for each output type is random and depends on the input type. We need to determine the purchase quantities of different types of input, before demands of different types of output are known. In our analytical results, we show that the expected total profit is jointly concave in the purchasing quantities and derive the optimality condition. Our multi-input and -output newsvendor model is suitable for processing industries in agriculture supply chain. In our numerical example, we apply our model to the rice milling industry, whose primary output is head rice and byproducts are broken rice, bran and husk. Our model can help the rice mill to decide which paddy types to procure and how much, in order to maximize the total expected profit from all outputs. We also show that the expected profit can be significantly better than using the standard newsvendor model.

1 INTRODUCTION

In a *newsvendor* (single-period inventory) model, there is a single opportunity to place an order before a random demand is known. Leftovers cannot be kept from one selling season to the next, due to obsolescence or perishability. The applications of the newsvendor model include a newspaper, a style good, a Christmas tree, bakery, and fresh produce. In a classical newsvendor model, there is only one product and no resource constraint. In a multi-item newsvendor model, the decision maker needs to decide the order quantities of multiple items before their random demands materialize, subject to several resource constraints. For instance, a news vendor, who manages a "newsstand," offering many different newspaper titles, needs to determine how many of each title to buy, in order to maximize the total expected profit from all titles, subject to a budget constraint and a limited newsstand size. We modify this multi-item model to include both multiple outputs and inputs: Each type of input has a different quality level and is processed into several types of outputs at different rates. The purchase price for the high-quality input is typically greater than the low-quality output, and the high-quality inputs gives a better yield. After processing, multiple outputs (e.g., a primary product and byproducts) are obtained and sold at different prices.

Our model is suitable for agricultural processing industries. The processing company needs to determine the purchase quantity for each type of input, before the random demands for all outputs are known, in order to maximize the total expected profit. In an agriculture supply chain, yield and crop quality are influenced by many factors including seeding time, harvest time, weather conditions, presence of nutrients within the soil at the farm location, and use of fertilizers and pesticides. An input type can be specified by yield and quality. For instance, in a rice processing industry, outputs from milling a paddy include head rice, broken rice, rice bran and husk (Wilasinee et al., 2010). A high-quality paddy gives a high yield of head rice. An input type can also specify other factors such as the percent impurity, the percent moisture content, the farm location, and the rice species. In a sugarcane milling process, the yield of raw sugar depends on the cane quality and the juice extraction efficiency. The supplier of sugarcane with the greater commercial cane sugar (CCS) receives the higher purchasing price (Higgins et al., 1998). In a vineyard harvesting problem, grape qualities depend on, e.g., the time of harvest, a wine block (a group of adjacent vineyard's fields growing the same vari-

Amaruchkul, K.
Newsvendor Model for Multi-Inputs and -Outputs with Random Yield: Applications to Agricultural Processing Industries.
DOI: 10.5220/0007346900720081
In *Proceedings of the 8th International Conference on Operations Research and Enterprise Systems (ICORES 2019)*, pages 72-81
ISBN: 978-989-758-352-0

ety of grapes), sugar and acidity levels determined by a winery's oenologist. The end-products can be classified into four different types, e.g., *Vins de Prestiges* and *Vins de Charme* (Arnaout and Maatouk, 2010). In many fresh fruit supply chains (e.g., apple, pear, banana, coconut), fruits are categorized into different grades, which usually depend on their sizes, among other factors. The low-quality farm with many aging trees and poor soil quality produces smaller sizes, which are bought at lower prices. Most agricultural raw materials need to be harvested at certain times of year. They are prone to deterioration after harvesting and processing. They are processed into different outputs before the demands for the outputs are known. The yield of the primary (most valuable) output depends on the crop quality. Thus, our multi-input and -output newsvendor model is applicable.

A literature review of the newsvendor (single-period inventory) model includes (Qin et al., 2011) and (Choi, 2012b). Multi-period inventory models are covered extensively in textbooks in inventory management, e.g., (Nahmias, 2009), (Silver et al., 1998), (Zipkin, 2000) and (Porteus, 2002). The multi-item newsvendor model was first formulated in (Hadley and Whitin, 1963): The order quantity of each item needs to be made in order to maximize the total expected profit subject to a single resource constraint. The budget constraint is a special case of the resource constraint. When the number of products is small, the exact solution can be obtained using a dynamic programming approach. When the number of products is large, several heuristics are proposed in, e.g., (Nahmias and Schmidt, 1984). (Moon and Silver, 2000) extends the multi-item newsvendor problem with a single resource constraint to include fixed ordering costs. (Lau and Lau, 1995) extends the multi-item newsvendor problem with a single resource constraint to multiple resource constraints. To contrast with the newsvendor problem, they refer to their multi-item multi-constraint model as the "newsstand problem." In the newsstand problem, there is no random yields; for instance, if we place an order of 50 units of the New York Times and 50 units of the Chicago Tribune, then all 50 units of the New York Times and 50 units of the Chicago Tribune are received, and they are put on the newsstand as long as the space constraint is satisfied. In our model, if the rice mill purchases 50 tons of paddy from a high-quality supplier and 50 tons of paddy from a low-quality supplier, then after the milling process, the total volume of head rice would be less than 100 tons, and a large fraction of the total head rice comes from the high-quality paddy. The multi-product multi-constraint newsvendor model is a convex optimization problem. Heuristic solutions based on a quadratic programming approach are presented in, e.g., (Abdel-Malek and Areeratchakul, 2007) and (Chernonog and Goldberg, 2018). Various extensions of the multi-production multi-constraint newsvendor model include an outsourcing strategy of shortages (Zhang and Du, 2010), an allocation decision on the raw material (Xie et al., 2018), a reservation policy to meet marketing needs (Chen and Chen, 2010), and carbon cap and trade mechanism (Zhang and Xu, 2013). The multi-product newsvendor problem is reviewed in (Turken et al., 2012) and (Choi, 2012a). In Section 1.5.2 in (Turken et al., 2012), "nearly all the models in this chapter assume single supplier.... It would be interesting to incorporate multiple suppliers into MPNP [multi-product newsvendor problem]." Different input types can be interpreted as different suppliers, who provide crops with different quality levels. To the best of our knowledge, our article is the first study of the newsvendor model with multiple inputs and outputs whose yields are random.

Operations research (OR) models applied to the agriculture supply chain are reviewed in (Kusumastuti et al., 2016), (Soto-Silva et al., 2016) and (Ahumada and Villalobos, 2009). The quality issue of the agricultural product is one of the most distinctive characteristics of the agriculture supply chain. This feature is explicitly incorporated in our multi-input and -output newsvendor model. The demand uncertainty and the random yield are other important characteristics of the agriculture OR models, and they are presented in (Tan and Comden, 2012) and (Kazaz, 2004). (Tan and Comden, 2012) studies a planning problem, in which the production quantity at each farm is random, depending on the planted area and the crop yield in that farm. In our model, different input types can be interpreted as different suppliers or farmers who provide different crop qualities. (Kazaz, 2004) formulates a two-stage stochastic programming problem for the production planning in the olive oil industry; the oil producer can buy olives from farmers at a unit cost varying with the yield. In our model, the unit cost also depends on the input type. The high-quality input type has a high purchasing cost, and the low-quality input type incurs a high pre-processing cost. This feature is also presented in these two papers. Nevertheless, they consider multiple farms (inputs) but a single product (output), whereas in ours there are both multiple inputs and outputs.

In this article, we assume that there are multiple types of perishable inputs and outputs. In contrast to the standard multi-item newsvendor, the number of inputs need not be equal to the number of outputs in our model. In the standard multi-item newsvendor model, there is a one-to-one correspondence between each input type and each output type. In the agricultural supply chain, the type of input usually represents the quality of the crop. The better type of input results in the larger yields of the high-value outputs. Furthermore, we allows the yield to be random. In most agricultural products, the yield tends to fluctuate. We develop the newsvendor for multiple inputs and outputs with random yields, and derive an optimal purchase quantity of each type of input in order to maximize the expected total profit from all outputs.

The rest of the paper is organized as follows: Section 2 presents the newsvendor model for multiple inputs and outputs with random yields, and the problem is analyzed in Section 3. We illustrate the application of our model to the rice processing industry and provide some managerial insights in Section 4. Finally, the conclusion is provided in Section 5.

2 FORMULATION

Consider a newsvendor model that produces n different outputs from m inputs. Let D_j be the random demand of output j. Let x_i be the purchase quantity of a type-i input. We need to decide $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ before knowing the demands $(D_1, D_2, \ldots, D_n)$. For each $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$, let $U_{ij}(x_i)$ be the random volume of a type-j output after processing x_i units of a type-i input. The volume of the type-j output depends not only the volume of the input but also the input type. For a concrete example, in the rice processing industry, the different moisture content paddies are categorized into different input types. The moist paddies with the high percentage of moisture content (%MC) would take longer to dry and result in a higher weight loss. For instance, a ton of paddy with 21% MC would incur a weight loss of 100 kilograms, whereas a ton of paddy with 30%MC would incur a weight loss of 200 kilograms (Thailand Deparment of Internal Trade, 2018).

Note that the decision $\mathbf{x}$ is taken before knowing output volume $U_{ij}(x_i)$ and consequently demand D_i. The probability distributions of volume outputs and demands are described as follows. After processing, the realization of output volume $U_{ij}(x_i, \omega)$ becomes known. Let Ω be the set of all *scenarios* associated with output volume. The triplet $(\Omega, \mathcal{A}, P)$ is a probability space, where $\mathcal{A}$ is an event and P is a probability. The expected volume of the type-j output after processing x_i units of a type-i input is given as

$$E[U_{ij}(x_i)] = \int_{\Omega} U_{ij}(x_i, \omega) P(d\omega)$$

where the above integral is a Lebesque integral. We assume that Ω is a finite set. Then, the random variable $U_{ij}(x_i)$ is discrete, and its expectation can be written as

$$E[U_{ij}(x_i)] = \sum_{\omega_\ell \in \Omega} U_{ij}(x_i, \omega_\ell) P(U_{ij}(x_i) = U_{ij}(x_i, \omega_\ell)).$$

Finally, the demand D_j becomes known. Assume that the demand D_j is a real-valued random variable and independent of the output vector $(U_{1j}, \ldots, U_{mj})$ for each j. Let F_j denote the distribution of demand D_j and F_j^{-1} be the corresponding quantile function. Note that the quantile function is strictly increasing on $[0,1]$. These assumptions are made to avoid technical difficulties, and they are not restrictive in practice. In strategic models like ours, the different scenarios Ω of the volume output might be obtained through experts' judgments, and in many situations there are only finitely possible outcomes (Birge and Louveaux, 1997).

Let $(y)^+ = \max\{y, 0\}$ denote the positive part of a real number y. Assume that the volume of a type-j output given the input $\mathbf{x}$ is defined as

$$Y_j(\mathbf{x}) = \sum_{i=1}^{m} U_{ij}(x_i). \tag{1}$$

The type-j sales are $S_j(\mathbf{x}) = \min(Y_j(\mathbf{x}), D_j)$, the leftovers are $W_j(\mathbf{x}) = (Y_j(\mathbf{x}) - D_j)^+$, and the shortages are $T_j(\mathbf{x}) = (D_j - Y_j(\mathbf{x}))^+$. The cost parameters are denoted by:

c_i = per-unit cost of a type-i input

p_j = per-unit selling price of a type-j output

h_j = per-unit salvage value of a type-j output

g_j = per-unit penalty cost for a type-j shortage.

The unit cost c_i includes both the purchasing cost and the pre-processing cost associated with the type-i input. The high-quality input typically has a large purchasing cost but a small pre-processing cost. Among all types of outputs, the primary output usually has the largest selling price. The penalty cost includes both a direct cost of the shortages and a loss of goodwill. The expected total profit is defined as follows:

$$\begin{aligned}
\pi(\mathbf{x}) &= \sum_{j=1}^{n} E\Big[p_j S_j(\mathbf{x}) + h_j W_j(\mathbf{x}) - g_j T_j(\mathbf{x})\Big] \\
&\quad - \sum_{i=1}^{m} c_i x_i \qquad (2)\\
&= \sum_{j=1}^{n} E[(p_j - h_j + g_j) \min(D_j, \sum_{i=1}^{m} U_{ij}(x_i))] \\
&\quad + \sum_{j=1}^{n} h_j \sum_{i=1}^{m} E[U_{ij}(x_i)] - \sum_{j=1}^{n} g_j E[D_j] \\
&\quad - \sum_{i=1}^{m} c_i x_i. \qquad (3)
\end{aligned}$$

In the first summation in (2), the first term is the expected revenue, and the second term is the expected salvage value from the leftovers from n output types. The last term is the total cost of all m input types. Equation (3) follows from the identity

$$\min(a,b) = a - (a-b)^+ \tag{4}$$

for any real numbers a and b. The first expectation in (3) is with respect to distributions of D_j and U_{ij}. Assume that we are risk neutral, and our objective is to maximize the expected total profit:

$$\max\{\pi(\mathbf{x}) : \mathbf{x} \geq \mathbf{0}\}. \tag{5}$$

Note that we allow the demand vector $(D_1,\ldots,D_n)$ to be dependent. Specifically, the expression (3) for the total expected profit remains valid, since the expectation of the sum of random variations is the sum of their expected values.

Our formulation subsumes the standard multi-product newsvendor model (Turken et al., 2012). Suppose that the number of inputs is equal to the number of outputs (i.e., $m = n$). There is a one-to-one correspondence between each input type and output type, and the yield rate is 100% for $i = j$ (i.e., $U_{ii}(x_i) = x_i$ and $U_{ij}(x_i) = 0$ for $i \neq j$). In other words, the type-i input is processed into only the type-i output. Then, the expected total profit becomes

$$\sum_{i=1}^{m} E[(p_i - h_i + g_i)\min(D_i,x_i) - (c_i - h_i)x_i - g_i D_i]. \tag{6}$$

Let $c_i^o = c_i - h_i$ be the *overage* cost, i.e., the expected per-unit cost of positive inventory remaining at the end of the period. Let $c_i^u = p_i - c_i + g_i$ be the *underage* cost, i.e., the per-unit cost of unsatisfied demand. Define the expected total cost as the sum of the expected overage and underage costs:

$$\sum_{i=1}^{m} E[c_i^o(x_i - D_i)^+ + c_i^u(D_i - x_i)^+]. \tag{7}$$

Using the identity (4) again, (7) becomes

$$\sum_{i=1}^{m} E[c_i^o x_i - (c_i^o + c_i^u)\min(D_i,x_i) + c_i^u D_i]. \tag{8}$$

Comparing (6) and (8), we see that minimizing the total expected cost is equivalent to maximizing the total expected profit. Our model reduces to the multi-product newsvendor model. The (unconstrained) optimal purchase quantity is given as

$$\begin{aligned} x_i^* &= F_i^{-1}(c_i^u/(c_i^u + c_i^o)) \qquad (9)\\ &= F_i^{-1}\left(\frac{p_i - c_i + g_i}{p_i - h_i + g_i}\right). \end{aligned}$$

The term $c_i^u/(c_i^u + c_i^o)$ is often referred to as the *critical ratio*. The optimal purchase quantity is the quantile, evaluated at the critical ratio.

3 ANALYSIS

Assume that $p_j - h_j + g_j \geq 0$ for each $j = 1,2,\ldots,n$. This assumption is not restrictive; in most practical cases, the per-unit selling price p_j exceeds the per-unit salvage value h_j, and the assumption holds. Further assume that

$$U_{ij}(x_i) = A_{ij}x_i \tag{10}$$

where a one-unit type-i input yields a random A_{ij}-unit type-j output. The assumption (10) states that the volume of a type-j output is linear in the volume of an input. The multiplicative form is extensively used in many applied operations research model (e.g., the linear programming model). For each input type i, each realization $(A_{i1}(\omega),A_{i2}(\omega),\ldots,A_{in}(\omega))$ satisfies $\sum_{j=1}^{n} A_{ij}(\omega) = 1$ and $A_{ij}(\omega) \geq 0$. An example of such distribution is a Dirichlet distribution, which is a multivariate generalization of the beta distribution. Note that a one-unit type-i input on average yields $E[A_{ij}]$ units of a type-j output:

$$E[A_{ij}] = \int_{\Omega} A_{ij}(\omega)P(d\omega), \tag{11}$$

where the integral in (11) is a Lebesque integral. For a discrete distribution on a sample space Ω, the expected yield (11) becomes

$$E[A_{ij}] = \sum_{\omega_\ell \in \Omega} A_{ij}(\omega_\ell)P\big(A_{ij} = A_{ij}(\omega_\ell)\big).$$

In Theorem 1, we show that the multi-input and -output newsvendor problem $\{\pi(\mathbf{x}) : x_i \geq 0\}$ is a convex programming problem.

Theorem 1. *The expected total profit $\pi(\mathbf{x})$ is jointly concave in $\mathbf{x}$.*

Proof. We will use the following result on convexity (page 529 in (Heyman and Sobel, 1984)). Let $g : \mathbb{R}^n \to \mathbb{R}$ be a concave function, and let $f : \mathbb{R} \to \mathbb{R}$ be a nondecreasing concave function. Then, the composite function $h : \mathbb{R}^n \to \mathbb{R}$ defined as $h(\mathbf{x}) = f(g(\mathbf{x}))$ is a concave function. For each realization ω, $f(y;\omega) = \min(D(\omega),y)$ is concave and nondecreasing, and $g(\mathbf{x};\omega) = \sum_{i=1}^{n} A_{ij}(\omega)x_i$ is linear (i.e., both concave and convex). From the assumption that $p_j - h_j + g_j \geq 0$, we have that

$$(p_j - h_j + g_j)\min(D_j(\omega), \sum_{i=1}^{m} A_{ij}(\omega)x_i)$$

is concave in $\mathbf{x}$ for each realization ω. The first term in (3) is concave, since the expectation of the concave function is also concave. The second term is linear under the multiplicative form (10). Thus, the expected total profit $\pi(\mathbf{x})$ is concave. $\square$

Theorem 1 shows that the objective function, the expected total profit, is concave. The constraint functions are linear. Thus, the multi-input and -output newsvendor problem (5) is convex programming. Algorithms to solve a convex programming problem are active research, and they can be classified into three groups, namely gradient algorithms, sequential unconstrained algorithms, and sequential-approximation algorithms (Hillier and Lieberman, 2005).

Concavity of the expected profit function is desirable, since we can guarantee that the first-order conditions provide the globally optimal solution. The optimality conditions are derived in Theorem 2.

Define the expected per-unit salvage of the type-i input as:

$$\begin{aligned} h'_i &= \int_\Omega \Big[\sum_{j=1}^{n} h_j A_{ij}(\omega)\Big] P(d\omega) \\ &= \sum_{j=1}^{n} h_j E[A_{ij}]. \end{aligned}$$

The expected per-unit price p'_i and penalty cost g'_i are defined similarly. Define the *expected overage cost* as $c^o_i = c_i - h'_i$. Define the *expected underage cost* as $c^u_i = p'_i - c_i + g'_i$. The *expected critical ratio* associated with the type-i input is

$$\xi_i = \frac{c^u_i}{c^u_i + c^o_i}.$$

Let $i^* = \operatorname{argmax}\{\xi_i\}$ be the input type with the largest critical ratio.

For short-hand, denote $c^s_j = p_j - h_j + g_j$.

Theorem 2. *An optimal purchase quantity that maximizes $\pi(\mathbf{x})$ is $x^*_k = 0$ for $k \neq i^*$ and $x^*_{i^*} > 0$ which satisfies*

$$\int_\Omega \Big[\sum_{j=1}^{n} c^s_j A_{i^*j}(\omega)\bar{F}_j(A_{i^*j}(\omega)x_{i^*})\Big] P(d\omega) = c^o_{i^*}. \qquad (12)$$

Proof. We want to maximize $\{\pi(\mathbf{x}) : \mathbf{x} \geq \mathbf{0}\}$. Recall the Kuhn-Tucker optimality conditions, there exist $\mu_k \geq 0$ and

$$\frac{\partial \pi(\mathbf{x})}{\partial x_k} + \mu_k = 0 \text{ for each } k = 1, 2, \ldots, n.$$

Then,

$$\begin{aligned} \frac{\partial \pi}{\partial x_i} &= 0 \text{ for } x_i > 0 \\ \frac{\partial \pi}{\partial x_i} &\leq 0 \text{ for } x_i = 0. \end{aligned}$$

The condition is also sufficient since $\pi(\mathbf{x})$ is concave (see Theorem 1).

Let $A_j = (A_{1j}, A_{2j}, \ldots, A_{mj})$. Note that

$$\begin{aligned} E[S_j(\mathbf{x})|A_j = A_j(\omega)] &= E[\min(D_j, \sum_{i=1}^{m} A_{ij}(\omega)x_i)] \\ &= \int_0^{\sum_{i=1}^{m} A_{ij}(\omega)x_i} \bar{F}_j(t)dt \end{aligned}$$

where the last equation follows from the tail-sum formula for expectation. Recall that

$$E[S_j(\mathbf{x})] = E[E[S_j(\mathbf{x})|A_j]].$$

Then,

$$E[S_j(\mathbf{x})] = \int_\Omega E[\min(D_j, \sum_{i=1}^{m} A_{ij}(\omega)x_i)] P(d\omega).$$

Taking the partial derivative with respect to x_i and using the Leibniz's rule, we have

$$\begin{aligned} \frac{\partial E[S_j(\mathbf{x})]}{\partial x_i} &= \frac{\partial}{\partial x_i} \int_\Omega E[\min(D_j, \sum_{k=1}^{m} A_{kj}(\omega)x_k)] P(d\omega) \\ &= \int_\Omega \frac{\partial}{\partial x_i} E[\min(D_j, \sum_{k=1}^{m} A_{kj}(\omega)x_k)] P(d\omega) \\ &= \int_\Omega \frac{\partial}{\partial x_i} \int_0^{\sum_{k=1}^{m} A_{kj}(\omega)x_k} \bar{F}_j(t)dt P(d\omega) \\ &= \int_\Omega A_{ij}(\omega)\bar{F}_j(\sum_{k=1}^{m} A_{kj}(\omega)x_k) P(d\omega) \end{aligned}$$

where the last equation follows from the fundamental theorem of calculus. From (3), the partial derivative of the expected profit $\partial\pi(\mathbf{x})/\partial x_k$ becomes

$$\begin{aligned} \frac{\partial \pi(\mathbf{x})}{\partial x_k} &= \sum_{j=1}^{n} c^s_j \int_\Omega A_{kj}(\omega)\bar{F}_j(\sum_{\ell=1}^{m} A_{\ell j}(\omega)x_\ell) P(d\omega) - c^o_k \\ &= \sum_{j=1}^{n} c^s_j \int_\Omega A_{kj}(\omega)\bar{F}_j(A_{i^*j}(\omega)x_{i^*}) P(d\omega) - c^o_k \end{aligned}$$

by construction of $\mathbf{x}^*$. Substituting the expression for $\partial\pi(\mathbf{x})/\partial x_k$, the optimality equations become

$$\sum_{j=1}^{n} c^s_j \int_\Omega A_{kj}(\omega)\bar{F}_j(A_{i^*j}(\omega)x_{i^*}) P(d\omega) - c^o_k + \mu_k = 0 \qquad (13)$$

For $k = i^*$, we have $x^*_k > 0$, $\mu_k = 0$, and (13) becomes (12). For $k \neq i^*$, (13) becomes

$$\mu_k = c^o_k - \sum_{j=1}^{n} c^s_j \int_\Omega A_{kj}(\omega)\bar{F}_j(A_{i^*j}(\omega)x_{i^*}) P(d\omega).$$

It follows from the definition of $i^* = \operatorname{argmax}\{\xi_i\}$ that $\mu_k \geq 0$ for $k \neq i^*$. The Kuhn-Tucker conditions hold. □

Theorem 2 allows us to reduce the problem size from m decision variables to a single decision variable, x_{i^*}. Several efficient one-dimensional search algorithms are available. We should purchase the "best" input type i^*, and the purchase quantity is given by (12).

Suppose that the number of inputs is identical to the number of outputs and the yield is 100%. Then, the optimality condition (12) reduces to

$$(p_i - h_i + g_i)\bar{F}_i(x_i) = c_i - h_i,$$

which is equivalent to the optimality condition in the multi-product newsvendor (9). Again, we see that our multi-input and -ouput newsvendor model becomes the standard multi-product newsvendor model.

Theorem 3 provides a sensitivity analysis: How would the optimal order quantity change, when some of the parameters change?

Theorem 3. *Assume that the "best" input type remains the same. The optimal order quantity $x^*_{i^*}$ is larger if one of the following conditions holds (*ceteris paribus*):*

1. *The per-unit selling price p_j increases.*
2. *The per-unit salvage value h_j decreases.*
3. *The per-unit penalty g_j increases.*
4. *The demand D_j is stochastically larger (in the usual stochastic order sense).*
5. *The per-unit cost c_i decreases.*

Proof. Note that the right-hand side (RHS) of the optimality equation (12)

$$\int_\Omega \Big[\sum_{j=1}^{n}(p_j - h_j + g_j)A_{i^*j}(\omega)\bar{F}_j(A_{i^*j}(\omega)x_{i^*})\Big]P(d\omega)$$

is decreasing in $x^*_{i^*}$. Furthermore, for a fixed $x^*_{i^*}$, the RHS increases, when the per-unit selling price p_j increases. Thus, the optimal order quantity becomes larger. The proof for the other cases is similar. □

The directional change in Theorem 3 makes economic sense. To determine which factors significantly affect the optimal order quantity and the optimal expected profit, we can create various sensitivity graphs, e.g., a *tornado diagram* and a *spider graph* (Hillier et al., 2000). The method to calculate the expected profit is illustrated in the next section.

4 NUMERICAL ILLUSTRATION

We apply our model to the rice processing industry in Thailand. Consider a rice mill which purchases paddy from middleman and farmers. The purchase price for paddy depends on the percent moisture content (%MC) and the head yield (HY). The paddy is classified into four groups of moisture content:

1. Group 15.0: %MC $\leq$ 15.0%
2. Group 19.9: %MC between 15.1–19.9%
3. Group 24.9: %MC between 20.0–24.9%
4. Group 29.9: %MC greater than 25.0%

Paddy with the %MC greater than 15% needs to be dried. The head yield can be divided into three groups, namely 40%, 45% and 50%. Table 1 shows the drying cost and the purchase price in Thai Baht (THB) per ton. For the same head yield, the purchase cost is higher for the paddy with the lower %MC. For the same %MC, the purchase cost is higher for the paddy with the larger head yield. In addition to the %MC and the head yield, the purchasing price depends on the rice varieties. The purchase prices in Table 1 are similar to the purchase prices of Thai aromatic rice paddy (KDML105); see (Thailand Deparment of Internal Trade, 2018).

Table 1: Purchase prices for paddy.

	HY			Drying	%
%MC	40	45	50	Cost	Loss
15.0	19600	20600	21600	0.00	0.0
19.9	18277	19210	20142	59.00	6.9
24.9	16807	17665	18522	62.10	14.4
29.9	15337	16120	16902	65.21	21.9

A type-i input is specified by the %MC and the HY. Table 2 shows $m = (4)(3) = 12$ input types. The per-unit cost of the type-i input, c_i, includes both the purchasing and drying costs. For instance, the type-2 input with %MC=19.9 and HY=40, and the drying cost is 59 THB/ton, so the per-unit cost $c_2 = 18277 + 59 = 18336$ THB/ton. In our example, we assume that a farmer brings paddies to the rice mill. In some other cases, the rice mill may travel to buy from the farmers whose produce paddies with high quality. Then, the input type would specify the %MC and the head yield as well as the location of the paddy field; the cost c_i may include the (round-trip) transportation from the rice mill to the field.

After the paddy is dried, a weight loss occurs, and the percent weight loss depends on the percent moisture content (see Table 1). The dried paddy is stored in a silo and waits for milling. The milling process consists of husk removing from paddy, whitening and polishing for bran removal, and finally separating head rice (the primary product) and broken rice into different grades. The milled rice is distributed to domestic and export markets. The rice bran can be sold to the rice bran oil industry. The husk is used

Table 2: Description of 12 input types.

i	%MC	HY	c_i
1	15.0	40	19600
2	19.9	40	18336
3	24.9	40	16869
4	29.9	40	15402
5	15.0	45	20600
6	19.9	45	19268
7	24.9	45	17727
8	29.9	45	16185
9	15.0	50	21600
10	19.9	50	20201
11	24.9	50	18584
12	29.9	50	16967

in paper production and biomass power generation. The average selling prices p_j, salvage values h_j and the penalty costs g_j (in THB/ton) for these $n = 4$ outputs (head rice ($j = 1$), broken rice ($j = 2$), rice bran ($j = 3$) and husk ($j = 4$)) are shown in Table 3. In Thailand, the selling price of head rice and broken rice are published online by the government (Thailand Deparment of Internal Trade, 2018). If the shortages of the head rice occur, then the rice mill tries to purchase the rice from other rice mills. The penalty cost g_1 includes the average purchasing cost, the transportation cost, and the loss of goodwill if the customer does not receive the entire rice volume in time. In our example, we assume that there are no penalty costs for the other byproducts, i.e., $g_j = 0$ for $j > 1$.

Assume that the demand for the type-j output, D_j, is independent and normally distributed with mean μ_j and standard deviation σ_j (in ton), as shown in Table 3. The coefficient of variation of demand is sufficiently small so that the probability of negative demand is negligible. (Our formulation in Section 2 and the analytical results in Section 3 hold without normality assumption.) Let ϕ denote the density function of the standard normal distribution and Φ denote the corresponding cumulative distribution function. The expected sales can be calculated easily:

$$E[\min(D_j, y_j)] = \mu_j - \sigma_j L\left(\frac{y_j - \mu_j}{\sigma_j}\right)$$

where the standard loss function is

$$L(z) = \phi(z) - z(1 - \Phi(z)).$$

Table 3: Selling prices, salvage values and penalty costs.

Output type (j)	Head 1	Broken 2	Bran 3	Husk 4
p_j	36800	12340	8999	1500
h_j	29872	5000	1000	500
g_j	1000	0	0	0
μ_j	94.00	64.00	53.00	24.00
σ_j	14.10	9.60	7.95	3.60

If the demand does not follow the normal distribution but other well-known distribution, then the expected sales can be calculated using the closed-form formula for the *limited expected value* (see, e.g., Table 12.2 in (Cunningham et al., 2008)). The demands for these four outputs are likely to be independent, since they are different products used in different industries, and they cannot be substituted for one another. In practice, we may have more output types. For instance, after being milled, the broken rice can be further graded into five levels, and their demands may be dependent. Our model can still be applied to the dependent outputs. The rice mill needs to determine the purchase quantities of different types of paddies, before the demands of all outputs are known.

Example 1. In our model, two sources of uncertainty are the demand and the random yield. For an illustrative purpose, we assume that there are 3 scenarios for the random yield, $\Omega = \{\omega_1, \omega_2, \omega_3\}$. The actual yields depend on the rice variety, the operations designed for rice kernel cracking and breakage, and the mechanical and physical properties (Correa et al., 2007). In stochastic programming approximations, we can find some relatively low cardinality discrete set of realizations that represents a good approximation of the true underlying distribution of the yield; details can be found in Chapter 9, (Birge and Louveaux, 1997). In scenario ω_ℓ, one unit (ton) of the type-i paddy results in $A_{i1}(\omega_\ell)$ units of head rice, $A_{i2}(\omega_\ell)$ units of broken rice, $A_{i3}(\omega_\ell)$ units of rice bran and $A_{i4}(\omega_\ell)$ units of rice husk (see Table 4). Assume that each scenario is equally likely to occur; i.e.,

$$P(A_{ij} = A_{ij}(\omega_\ell)) = 1/3$$

for each $\ell = 1, 2, 3$. For each input type and for the type-1 output (head rice), the yield in scenario ω_2 (resp., ω_3) is approximately 10–20% higher (resp., lower) than the yield in scenario ω_1. The higher the head yield, the higher yield for head rice. The paddy with the higher moisture content incurs the larger percent weight loss.

Note that for the paddy with the %MC less than 15% (i.e., $i \in \{1, 5, 9\}$), there is no weight loss, and $\sum_{j=1}^{4} A_{ij}(\omega_\ell) = 1$. For the other types, $\sum_{j=1}^{4} A_{ij}(\omega_\ell) < 1$. Suppose that we define the type-5 output to be the processing loss. Let $A_{i5} = 1 - \sum_{j=1}^{4} A_{ij}$. Then, the output vector $U_{ij}(x_i) = A_{ij} x_i$ now has a valid probability distribution, i.e., $\sum_{j=1}^{5} A_{ij} = 1$ and $A_{ij} \geq 0$.

To use the optimality condition (12), we first find the "best" type of input by ranking the expected critical ratio. From Table 5, we find that the most profitable input type is $i^* = 10$ (%MC=19.9 and HY=50). The ranking in Table 5 also helps the rice mill select a paddy supplier. Note that the paddy types with the low head yield (i.e., $i \in \{1,2,3,4\}$) are ranked among the lowest. It is more important to focus on the head yield, compared to the moisture content, since it is relatively cheap to dry the paddy. This finding is consistent with the result found in the agricultural academic journal (Wilasinee et al., 2010).

Upon solving the optimality equation (12), we find that the optimal purchase quantity is $x_{10}^* = 187$ ton. The total purchasing and drying costs is 3781345 THB. The expected revenue is 3820335 THB, and the expected profit is 125331 THB. The expected revenues and profits for all scenarios are shown in Table 6.

We see that the profit is very sensitive to the output yield of the milling process. In scenario 2, the head rice yield is 10–20% higher than the yield in scenario 1, but the expected profit in scenario 2 is double. Our model can also be used to quantify the benefit from milling process improvement. $\square$

Example 2. We want to highlight the benefit of our model which includes multiple outputs (i.e., the head rice as the primary product and all byproducts). In particular, we will identify the set of parameters such that our model significantly outperforms the standard newsvendor model. For simplicity, assume that $\Omega = \{\omega_1\}$ and that there are no penalty costs $g_j = 0$ for all j.

Table 4: Yields of the four outputs $A_{ij}(\omega_\ell)$.

ℓ	i	$A_{i1}(\omega_\ell)$	$A_{i2}(\omega_\ell)$	$A_{i3}(\omega_\ell)$	$A_{i4}(\omega_\ell)$
1	1	0.4000	0.1976	0.2541	0.1483
1	2	0.3724	0.1840	0.2366	0.1380
1	3	0.3424	0.1692	0.2175	0.1269
1	4	0.3124	0.1544	0.1985	0.1158
1	5	0.4500	0.1812	0.2329	0.1359
1	6	0.4190	0.1687	0.2169	0.1265
1	7	0.3852	0.1551	0.1994	0.1163
1	8	0.3515	0.1415	0.1819	0.1061
1	9	0.5000	0.1647	0.2118	0.1235
1	10	0.4655	0.1533	0.1972	0.1150
1	11	0.4280	0.1410	0.1813	0.1057
1	12	0.3905	0.1286	0.1654	0.0965
2	2	0.4096	0.1468	0.2366	0.1380
2	3	0.3664	0.1452	0.2175	0.1269
2	4	0.3218	0.1450	0.1985	0.1158
2	5	0.4590	0.1722	0.2329	0.1359
2	6	0.4358	0.1519	0.2169	0.1265
2	7	0.3929	0.1474	0.1994	0.1163
2	8	0.3796	0.1134	0.1819	0.1061
2	9	0.5100	0.1547	0.2118	0.1235
2	10	0.4934	0.1254	0.1972	0.1150
2	11	0.4323	0.1367	0.1813	0.1057
2	12	0.4022	0.1169	0.1654	0.0965
3	1	0.3920	0.2056	0.2541	0.1483
3	2	0.3650	0.1914	0.2366	0.1380
3	3	0.3424	0.1692	0.2175	0.1269
3	4	0.2968	0.1700	0.1985	0.1158
3	5	0.4230	0.2082	0.2329	0.1359
3	6	0.3813	0.2064	0.2169	0.1265
3	7	0.3852	0.1551	0.1994	0.1163
3	8	0.3515	0.1415	0.1819	0.1061
3	9	0.4600	0.2047	0.2118	0.1235
3	10	0.4469	0.1719	0.1972	0.1150
3	11	0.4237	0.1453	0.1813	0.1057
3	12	0.3749	0.1442	0.1654	0.0965

Table 5: Expected overage and underage costs and the ranking based on the expected critical raio.

i	%MC	HY	c_i^o	c_i^u	ξ_i	Rank
1	15.0	40	6269	535	0.079	11
2	19.9	40	5739	600	0.095	9
3	24.9	40	5315	513	0.088	10
4	29.9	40	5093	219	0.041	12
5	15.0	45	6100	793	0.115	7
6	19.9	45	5802	616	0.096	8
7	24.9	45	5123	782	0.132	6
8	29.9	45	4509	883	0.164	5
9	15.0	50	5816	1169	0.167	4
10	19.9	50	5197	1313	0.202	1
11	24.9	50	4860	1124	0.188	2
12	29.9	50	4478	981	0.180	3

Table 6: Expected profit.

Scenario	1	2	3
Head rice sales	84.29	87.52	81.75
Revenue	3819035	3873344	3768627
Profit	113336	230482	32176

Suppose that the decision maker does not consider multiple outputs and focuses only on the single primary output, which is the head rice (i.e., $n = 1$ and $j = 1$). Then,

$$p_i' = p_i E[A_{i1}] = p_i A_{i1}(\omega_1)$$
$$h_i' = p_i E[A_{i1}] = h_i A_{i1}(\omega_1).$$

When the decision maker considers only one input ($n = 1$), let $i^*(n = 1)$ be the best input type and $\xi_{i^*(n=1)}$ the corresponding critical ratio. The optimality condition (9) reduces to

$$\begin{aligned} x_{i^*(n=1)} &= A_{i1}(\omega_1) F_1^{-1}(\xi_{i^*(n=1)}) \\ &= A_{i1}(\omega_1)[\mu_1 + \sigma_1 \Phi^{-1}(\xi_{i^*(n=1)})] \end{aligned} \tag{14}$$

if the critical ratio $\xi_{i^*(n=1)} > 0$; otherwise the purchase quantity is zero.

We will vary the selling price of the primary input from 400000 to 70000 THB. Suppose that the selling price of the head rice is large, say $p_1 = 65000$. Then, the type-9 input has the largest critical ratio. Based on (14), the decision maker purchases 197 tons of the type-9 paddy and obtains the expected profit of 2667857. When we consider all output types (i.e., $n = 4$), an optimal purchase quantity is 215 tons, and the optimal expected profit is 2702104; the percent profit loss is $(2702104 - 2667857)/2702104 = 1.27\%$. On

Table 7: Profit loss from ignoring byproducts.

p_1	Purchase quantity $n=1$	$n=4$	Opt. profit	%Δ profit
40000	0	186	415232	100.00
45000	155	198	862410	11.35
50000	176	205	1317887	4.32
55000	186	209	1777279	2.52
60000	192	213	2238948	1.71
65000	197	215	2702104	1.27
70000	200	218	3166306	0.99

the other hand, suppose that the selling price of head rice is small as in the previous example, $p_1 = 36800$. Then,

$$p'_9 = p_1 A_{9,1}(\omega_1) = 18400 < 21600 = c_9,$$

and consequently $\xi_9 < 0$, the decision maker that ignores all byproducts would not buy any paddies. Table 7 shows the purchase quantity when the decision maker focuses only on the head rice ($n = 1$), the optimal purchase quantity when all output types ($n = 4$) are considered, the corresponding optimal expected profits, and the percent profit loss, as the selling price of the head rice varies. The purchase quantity when the decision maker ignores all byproducts is less than the optimal quantity. The difference between the two quantities becomes smaller when the selling price of the head rice becomes larger. In other words, our model outperforms the standard newsvendor model when the processing firm has a significant revenue potential from the other products (other than the primary product). We can perform an ABC classification on all outputs to determine their importance. If the volumes of some byproducts are significant or their selling prices are not negligible, the processing firm could benefit from using our model with multiple outputs. □

5 CONCLUSION

In summary, we formulate the multi-input and -output newsvendor model with the random yield. The optimization problem of determining the purchase quantity of each input type is shown to be a convex programming problem. In the numerical example, we use the optimality condition to find the optimal purchase quantities of different types of paddy for the rice mill, who wants to maximize the total expected profit from the head rice and its byproducts, namely the broken rice, bran and husk.

Some future research directions are identified below. In the standard multi-product newsvendor model, the resource constraints are considered. We could extend our model to include the resource constraints. Obtaining some structural results and deriving the optimality condition for the multi-input and -output newsvendor model with resource constraints are interesting from a theoretical viewpoint. From a practical viewpoint, we could consider price-dependent demands. In many agricultural products, the selling price itself is random, depending on various factors such as the world economic output, the global production and consumption. In our current model, the selling price is deterministic, and the demand distribution is given exogenously. A stochastic programming framework can be used to capture the price-dependent demand. Another extension is to consider contract farming, one of the most common arrangements between a farmer and a private agricultural processing company, wherein the farmer agrees to produce at a pre-agreed market price for procurement by the other party. A mechanism design framework can be applied in order to improve efficiency of the agriculture supply chain through an optimal contract. Finally, when the variability of yield is significant, and the decision maker is no longer risk neutral, the objective of maximizing the expected profit would not be suitable. Instead, one could maximize the expected utility, in which risk is explicitly taken into account. We hope to pursue these or related issues in the future.

ACKNOWLEDGEMENTS

The problem was materialized after some discussions with Mr. Chatbodin Sritrakul, our part-time master student who owns a rice mill in the Northeast of Thailand. His independent project, a part of requirement for a master degree in logistics management at the school, was related to our model.

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