The Dynamics of Narrow-minded Belief

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Abstract: The purpose of this paper is to consider and formalize an important factor of human intelligence, belief affected by passion, which we call narrow-minded belief. Based on Public Announcement Logic, we define our logic, Logic Of Narrow-minded belief (LON), as that which includes such belief. Semantics for LON is provided by the Kripke-style semantics, and a proof system for it is given by the Hilbert-style proof system. We then provide a proof of the semantic completeness theorem for the proof system with the innermost strategy of reducing a formula for LON. Using LON, we formally analyze the mental state of the hero of Shakespeare's tragedy *Othello* as an example of narrow-minded belief and its formalization.

1 INTRODUCTION

Love is blind, and hatred is also blind. To generalize these phrases, we may say that passion causes narrow-mindedness. It is not unusual that people cannot emotionally stop believing what they do not want to believe without any specific reason to believe so. The hero of William Shakespeare's play, *Othello*, is involved in a pitiful but possible situation where he wants to believe his wife's chastity but he cannot since he heard a bad rumor about her. It may be difficult to answer whether or not he believes that his wife is a betrayer of their marriage given that he has heard this rumor. In this situation, Othello has at least two different types of belief and/or knowledge. One is *passionate* or *narrow-minded belief*, which he is willing to believe or cannot stop believing emotionally. The other is belief, which is more rational (less passionate) or, without considering any philosophical discussions regarding the relationship between knowledge and belief, it may even be said, is *knowledge* whereby he judges something based on information attained via rational inferences. The latter type of knowledge or belief is treated by a standard epistemic (or doxastic) logic and the current researchers would like to introduce the former belief, passionate belief or narrow-minded belief.

In fact, the notion of passion has a philosophically and psychologically profound meaning in terms of belief, and it is highly possible that such emotional belief plays a significant role in rationality. In *A Treatise of Human Nature*, Hume famously (or even notoriously) wrote the following quotation.

> [T]he principle, which opposes our passion, cannot be the same with reason, and is only called so in an improper sense. We speak not strictly and philosophically when we talk of the combat of passion and of reason. Reason is, and ought only to be the slave of the passions [...]. (Hume, 1739, Book II, Sec. 3, Part 3).

Here, Hume says not only that passion has the same significance as rationality, but also that reason is a subordinate of passion. We introduce one more quotation from modern literature, Damasio's *Descartes' error*, to support the importance of consideration on the relationship between passion and rationality.

> [T]here may be a connecting trail, in anatomical and functional terms, from reason to feelings to body. It is as if we are possessed by a passion for reason [...]. Reason, from the practical to the theoretical, is probably constructed on this inherent drive by a process which resembles the mastering of a skill or craft. Remove the drive, and you will not acquire the mastery. But having the drive does not automatically make you a master. (Damasio, 1994, Part III, Chap. 11)

By referring neurological evidence, Damasio argues that feeling (or passion) and rationality are strongly connected with other, and they cannot be separated

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as Descartes thought. The current researchers would like to take a similar stance to that of Damasio, where passion and rationality (in our term, narrow-minded belief and knowledge) are related to one another in a formal language of epistemic logic.

In this paper, we treat such a paradigm of agent communication that each agent changes his/her belief, after receiving messages from others, to strengthen/ weaken his/her tolerance. Towards this motivation, we present a logic that adequately reflects human minds which tends to be biased by certain kinds of information.

The outline of the paper is as follows. In Section 2, we introduce logic of narrow-minded belief (LON) which is based on Public Announcement Logic by Plaza (Plaza, 1989) and refers to the ideas of explicit and implicit belief in dynamic epistemic awareness logic by van Benthem & Velázquez-Quesada (van Benthem and Velázquez-Quesada, 2010). Its semantics are given by an expansion of the Kripke-style semantics. In Section 3, we attempt to investigate and formalize a person's belief and emotion through focusing on a literary work, *Othello* since this is a story of delicate transition of the hero's narrow-minded belief towards his wife. In Section 4, we introduce a Hilbert-style proof system LON of LON, and some proof theoretic properties. In Section 5, we give a proof of the semantic completeness of our proof system LON (Theorem 5.4) through the innermost strategy for reducing a formula for LON into a formula without announcement operators. In Section 6, we introduce related epistemic/doxastic logics to the present work.

2 LANGUAGE AND SEMANTICS OF LON

2.1 Language

First of all, we address the syntax of LON. Let $Atom = \{p, q, \ldots\}$ be a countable set of atomic propositions. Then, formula φ of the language $\mathcal{L}_{(\mathbf{KN}\ominus\oplus!)}$ is inductively defined as follows ($p \in Atom$):

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \to \varphi) \mid \mathbf{K}\varphi \mid \mathbf{N}\varphi \mid [\ominus\varphi]\varphi \mid [\oplus\varphi]\varphi \mid [\,!\,\varphi]\varphi.$$

We define other Boolean connectives such as $\varphi \wedge \chi$, $\varphi \vee \chi$, $\varphi \leftrightarrow \chi$ and $\bot$ in a usual manner. We call operators $[\oplus\varphi], [\ominus\varphi]$ and $[\,!\,\varphi]$ *announcement operators*. Besides, $\widehat{\mathbf{K}}$ is defined by $\neg\mathbf{K}\neg$ and $\widehat{\mathbf{N}}$ is defined by $\neg\mathbf{N}\neg$. Note that $\mathbf{K}$ and $\mathbf{N}$ can be considered as the box operator $\Box$ in modal logic, and $\widehat{\mathbf{K}}$ and $\widehat{\mathbf{N}}$ can be considered as the diamond operator $\Diamond$.

- $\mathbf{K}\varphi$ reads 'the agent knows that φ',
- $\mathbf{N}\varphi$ reads 'the agent narrow-mindedly believes that φ',
- $[\ominus\varphi]\chi$ reads 'after obtaining information φ which may *strengthen* the agent's narrow-mindedness, χ holds',
- $[\oplus\varphi]\chi$ reads 'after obtaining information φ which may *weaken* the agent's narrow-mindedness, χ holds,' and
- $[\,!\,\varphi]\chi$ reads 'after obtaining truthful information (announcement) φ, χ holds'.

2.2 Semantics

Let us go on to the semantics of LON. We call the tuple $\langle S, R, V\rangle$ an *epistemic model* (or *a Kripke model*) if the domain S is a nonempty set of states, the accessibility relation R is an equivalence relation on S and $V : Atom \to \mathcal{P}(S)$ is a valuation function. The set S is called domain of $\mathcal{M}$ and may be denoted by $\mathcal{D}(\mathcal{M})$. Subsequently, we define an *epistemic narrow-doxastic model* (or simply *en-model*) $\mathcal{M} = \langle S, R, Q, V\rangle$ where the components of S, R and V are the same as that of the epistemic model, and Q is a binary relation on S such that $Q \subseteq R$.

Definition 2.1 (Satisfaction relation)**.** *Given an en-model $\mathcal{M}$, a state $s \in \mathcal{D}(\mathcal{M})$, and a formula $\varphi \in \mathcal{L}_{(\mathbf{KN}\ominus\oplus!)}$, we define* the satisfaction relation $\mathcal{M}, s \models \varphi$ *as follows:*

$$\begin{array}{lll}
\mathcal{M}, s \models p & \textit{iff} & s \in V(p), \\
\mathcal{M}, s \models \neg\varphi & \textit{iff} & \mathcal{M}, s \not\models \varphi, \\
\mathcal{M}, s \models \varphi \to \chi & \textit{iff} & \mathcal{M}, s \models \varphi \textit{ implies } \mathcal{M}, s \models \chi, \\
\mathcal{M}, s \models \mathbf{K}\varphi & \textit{iff} & \textit{for all } x \in S : sRx \textit{ implies } \mathcal{M}, x \models \varphi, \\
\mathcal{M}, s \models \mathbf{N}\varphi & \textit{iff} & \textit{for all } x \in S : sQx \textit{ implies } \mathcal{M}, x \models \varphi, \\
\mathcal{M}, s \models [\ominus\varphi]\chi & \textit{iff} & \mathcal{M}^{\ominus\varphi}, s \models \chi, \\
\mathcal{M}, s \models [\oplus\varphi]\chi & \textit{iff} & \mathcal{M}^{\oplus\varphi}, s \models \chi, \\
\mathcal{M}, s \models [\,!\,\varphi]\chi & \textit{iff} & \mathcal{M}, s \models \varphi \textit{ implies } \mathcal{M}^{!\varphi}, s \models \chi,
\end{array}$$

where the notations $\mathcal{M}^{\ominus\varphi}$, $\mathcal{M}^{\oplus\varphi}$ and $\mathcal{M}^{!\varphi}$ above respectively indicate the en-models defined by $\mathcal{M}^{\ominus\varphi} = \langle S, R, Q^{\ominus\varphi}, V\rangle$, $\mathcal{M}^{\oplus\varphi} = \langle S, R, Q^{\oplus\varphi}, V\rangle$ and $\mathcal{M}^{!\varphi} = \langle [\![\varphi]\!]_{\mathcal{M}}, R^{!\varphi}, Q^{!\varphi}, V^{!\varphi}\rangle$ with

$$\begin{array}{lll}
[\![\varphi]\!]_{\mathcal{M}} & := & \{x \in S \mid \mathcal{M}, x \models \varphi\}, \\
R^{!\varphi} & := & R \cap [\![\varphi]\!]_{\mathcal{M}} \times [\![\varphi]\!]_{\mathcal{M}}, \\
Q^{\ominus\varphi} & := & Q \cup \{(s,t) \in R \mid s \in [\![\varphi]\!]_{\mathcal{M}} \textit{ or } t \in [\![\varphi]\!]_{\mathcal{M}}\}, \\
Q^{!\varphi} & := & Q \cap [\![\varphi]\!]_{\mathcal{M}} \times [\![\varphi]\!]_{\mathcal{M}}, \\
Q^{\oplus\varphi} & := & Q \cap S \times [\![\varphi]\!]_{\mathcal{M}}, \\
V^{!\varphi}(p) & := & V \cap [\![\varphi]\!]_{\mathcal{M}} \textit{ (where } p \in Atom\textit{)}.
\end{array}$$

Then we define the validity of a formula in a usual way.

Definition 2.2 (Valid). *A formula* φ *is* valid at $\mathcal{M}$ *if* $\mathcal{M}, s \models \varphi$ *for any* $s \in \mathcal{D}(\mathcal{M})$*, and we write* $\mathcal{M} \models \varphi$*. A formula* φ *is* valid *if* $\mathcal{M} \models \varphi$*, for any en-model* $\mathcal{M}$*, and we write* $\models \varphi$*.*

We confirm that an en-model, which is modified by announcement operators $[\oplus\varphi]$, $[\ominus\varphi]$ and $[\,!\,\varphi]$, preserves frame properties, i.e., R is an equivalence relation and the subset relation $Q \subseteq R$.

Proposition 2.1 (Preserving frame properties). *Let* $\varphi \in \mathcal{L}_{(\mathbf{KN}\ominus\oplus!)}$ *be any formula. If* $\mathcal{M} = \langle S, R, Q, V\rangle$ *is an en-model, then* $\mathcal{M}^{\ominus\varphi} = \langle S, R, Q^{\ominus\varphi}, V\rangle$*,* $\mathcal{M}^{\oplus\varphi} = \langle S, R, Q^{\oplus\varphi}, V\rangle$ *and* $\mathcal{M}^{!\varphi} = \langle [\![\varphi]\!]_{\mathcal{M}}, R^{!\varphi}, Q^{!\varphi}, V^{!\varphi}\rangle$ *are also en-models.*

Proof. What we wish to show is that (1) $R^{!\varphi}$ is an equivalence relation (i.e., it satisfies reflexivity, Euclidicity and transitivity), and (2) the subset relation $Q^{!\varphi} \subseteq R^{!\varphi}$, (3) the subset relation $Q^{\ominus\varphi} \subseteq R$ and (4) the subset relation $Q^{\oplus\varphi} \subseteq R$. We only treat one of (1) and (3) in the following.

(1)-2 $R^{!\varphi}$ **satisfies Euclidicity.** Fix any $x, y, z \in [\![\varphi]\!]_{\mathcal{M}}$. Suppose $xR^{!\varphi}y$ and $xR^{!\varphi}z$, and show $yR^{!\varphi}z$. Since R is Euclidean i.e., xRy and xRz jointly imply yRz for all $x, y, z \in S$. By the assumption $x, y, z \in [\![\varphi]\!]_{\mathcal{M}} \subseteq S$, we have $x, y, z \in S$ and yRz. so we get the goal $R^{!\varphi}$ is also Euclidean with $y, z \in X$.

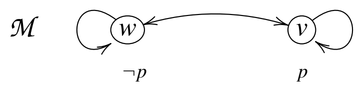
(3) Fix any $(x, y) \in Q \cup \{(x, y) \in R \mid x \in [\![\varphi]\!]_{\mathcal{M}}$ or $y \in [\![\varphi]\!]_{\mathcal{M}}\}$, and we show $(x, y) \in R$. Suppose $(x, y) \in Q$. Then we obtain $(x, y) \in R$ with $Q \subseteq R$. Suppose $(x, y) \in \{(x, y) \in R \mid x \in [\![\varphi]\!]_{\mathcal{M}}$ or $y \in [\![\varphi]\!]_{\mathcal{M}}\}$. So, we obtain $(x, y) \in R$.

□

3 EXAMPLES OF FORMALIZATION OF NARROW-MINDED BELIEF

3.1 Comments of Knowledge and Narrow-minded Belief Operators

Before moving on the topic of narrow-minded belief, we add some comments on the general features of knowledge operator $\mathbf{K}$ and accessibility relation R in epistemic logics. Let us look at the epistemic model $\langle S, R, V\rangle = \langle\{w, v\}, S^2, V\rangle$ where $V(p) = \{v\}$ (that can be regarded as an en-model $\mathcal{M} = \langle\{w, v\}, S^2, \varnothing, V\rangle$), and the graphic form of this model is as follows.



In this model, at world w, the agent is ignorant about p's truth-value. This is because the formula $\widehat{\mathbf{K}}p \wedge \widehat{\mathbf{K}}\neg p$, which intuitively means that the agent does not know whether p, is true at w. As it implies, in epistemic logic, an arrow between states has a negative meaning in general. In other words, van Ditmarsch et al. state that "the more worlds an agent considers possible, the less he believes, and vice versa." (van Ditmarsch et al., 2008, p.55). The operator $\widehat{\mathbf{K}}$ represents at least one arrow in an epistemic model. The narrow-minded belief operator $\widehat{\mathbf{N}}$ basically preserves these features; nevertheless, we cannot say that 'the more worlds an agent considers possible, the less he believes, and vice versa' in case of the operator $\mathbf{N}$ since the narrow-mind belief is affected by uncertain information or even the agent's imagination and may be wrong. In other words, to express such capricious belief, we introduce the operator $\mathbf{N}$.

Additionally, we note on the frame property of R and Q. The accessibility relation R represents the accessibility relation for knowledge, and so we assume that the agent is an introspective agent, i.e., R is an equivalence relation. Moreover, the formulas of $\mathbf{K}\varphi \to \varphi$, $\mathbf{K}\varphi \to \mathbf{KK}\varphi$ (positive introspection) and $\neg\mathbf{K}\varphi \to \mathbf{K}\neg\mathbf{K}\varphi$ (negative introspection) are valid at $\mathcal{M}$ where its accessibility relation is equivalence relation. However, since Q represents a narrow-minded belief, we do not assume the agent is introspective since introspectiveness is based on some kind of rationality, which is the exact opposite of narrow-mindedness. That is why Q does not have any frame property. By distinguishing these two accessibility relations, we formally express the distinction between knowledge and narrow-minded belief.

3.2 Formalizing Othello's Narrow-minded Belief

As mentioned in the introduction, our target, which we consider and formalize, is Shakespeare's *Othello* as it depicts a typical case of the change in a person's delicate mental state. Its story depicts how the lives of the four main characters (Othello, Desdemona, Iago and Cassio) are woven together and driven by passion. The following is the short summary of the play:

> General in the Venetian military Othello was recently married to a rich senator's daughter Desdemona. Although there is a great disparity of age between the two, they build a good relationship of trust, and Othello and Desdemona love and believe each other from their

hearts. However, Othello's trusted subordinate Iago who secretly holds a deep grudge against Othello tells him a rumor that Desdemona is having an affair with a young handsome soldier named Cassio. This causes Othello to feel uncertain towards his wife's innocence. Deepening Othello's doubt against his wife, Iago steals Desdemona's handkerchief, a present from Othello, and leads Cassio up to find it. Using the handkerchief as proof, Iago succeeds in convincing Othello that Desdemona has engaged in an immoral relationship with Cassio. Finally, Othello narrow-mindedly believes what Iago has told him and he feels great jealousy and anger towards his wife. Even though Desdemona protests her innocence, Othello, who is now mad with jealousy, kills his wife in a fit of passion. Following her death, Desdemona's servant confesses that her mistress was innocent and that Iago fabricated the story, which resulted in such a tragedy. Othello comes to his senses and realizes his mistake, at which point he loses hope and takes his own life.

Of course, this summary is extremely simplified and actual tale is more intricately woven. There are at least four main scenes in the story, which highlight Othello's narrow-minded belief, and we would like to focus on these in this paper. The four main points are as follows:

1. Othello believes Desdemona from the heart.
2. Iago spreads a bad rumor about Desdemona, which causes doubt about her innocence in Othello's mind.
3. Iago uses fake evidence (a handkerchief) to convince Othello of Desdemona's immoral actions and he narrow-mindedly believes it.
4. A servant truthfully informs Othello that Desdemona is innocent.

Othello's mind, including narrow-minded belief in each of the four scenes, may be semantically modeled as follows. We note that, in the graphic form of en-models, the double circle indicates the actual state. In addition, arrows of the straight line represent the line of R and arrows of the dotted line represent that of Q. Moreover, let an atomic proposition p to read 'Desdemona is having an affair,' and $Atom = \{p\}$.

(1) Othello deeply believes his wife. In the initial stage, Othello, who was recently married, believes his wife from the depth of his heart and does not doubt her immorality. However, Othello does not have any specific evidence that Desdemona is having an affair and he does not actually know if she is innocent or not at this stage. Therefore, the initial stage already includes some contradiction in his mind, i.e., he does not explicitly know if she is innocent, but he narrow-mindedly believes her. Thus, the mental state of Othello at the opening of the play may formally be expressed by en-model $\mathcal{M} = \langle S, R, Q, V\rangle = \langle\{s,t\}, S^2, \{(s,s)\}, \{p \mapsto \{t\}\}\rangle$. Therefore, we may say that, at this stage, formulas $\widehat{\mathbf{K}}p \wedge \widehat{\mathbf{K}}\neg p$ and $\mathbf{N}\neg p$ are valid at $\mathcal{M}$.

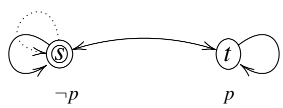


Figure 1: $\mathcal{M}$.

(2) Iago spreads a bad rumor about Desdemona, which leads to doubts in Othello's mind. After Iago tells Othello a bad rumor ($[\ominus p]$) about Desdemona, he begins to doubt his wife. In other words, he is now unsure about her constancy and does not know if she is innocent or not. Separately from Othello's narrow-minded belief, his state of knowledge remains unchanged since he has not obtained any new truthful information and can only go by Iago's story in which his wife is accused of infidelity. Then the mental state of Othello at the second stage of the play may formally be expressed by en-model $\mathcal{M}^{\ominus p} = \langle S, R, Q^{\ominus p}, V\rangle = \langle\{s,t\}, S^2, S^2, \{p \mapsto \{t\}\}\rangle$, where the formula $\widehat{\mathbf{N}}p \wedge \widehat{\mathbf{N}}\neg p$ is now valid at this en-model. This formula represents a confusion in his mind about his wife's innocence.

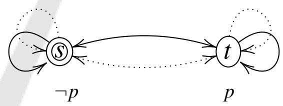


Figure 2: $\mathcal{M}^{\ominus p}$.

(3) Iago uses fake evidence to convince Othello of Desdemona's immorality. At this stage of the play, Iago attempts to deceive his superior, Othello, by using fake evidence (Desdemona's handkerchief) to pretend she spent her time with Cassio, and Othello is completely taken in. Consequently, Othello completely loses his self-control, and strongly and narrow-mindedly believes that his wife is having an affair with Cassio. This is also represented by en-model $\mathcal{M}^{\ominus p \oplus p} = \langle S, R, Q^{\ominus p \oplus p}, V\rangle = \langle\{s,t\}, S^2, \{(t,t),(s,t)\}, \{p \mapsto \{t\}\}\rangle$. Formally, in his mind, the formula $\widehat{\mathbf{N}}p$ is valid at this en-model, but $\widehat{\mathbf{N}}\neg p$ is not anymore. Let us remind the reader that in

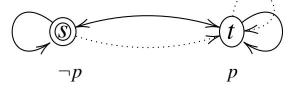


Figure 3: $\mathcal{M}^{\ominus p \oplus p}$.

the case of the operator $\widehat{\mathbf{N}}$, it does *not* mean that if the number of arrows is reduced, then the agent's ignorance is reduced.

(4) A servant truthfully informs that Desdemona is innocent. In the last scene of the play, Desdemona's faithful servant truthfully tells the fact that Desdemona is innocent, implying that Othello's narrow-minded belief regarding his wife is completely erroneous. Othello faces such a surprising fact and he is heart-broken by the confession. This is represented by en-model $\mathcal{M}^{\ominus p \oplus p ! \neg p} = \langle [\![\neg p]\!]_{\mathcal{M}^{\ominus p \oplus p}}, R^{!\neg p}, Q^{\ominus p \oplus p ! \neg p}, V^{!\neg p} \rangle = \langle \{s\}, \{(s,s)\}, \emptyset, \emptyset \rangle$. Formally, by the truthful information of $\neg p$, a state t where p holds is eliminated, and as a result, while the agent (Othello) knows $\neg p$ (his wife is innocent), the arrow of narrow-minded belief is empty. This means that he narrow-mindedly believes everything even if it is a contradiction $\mathcal{M}^{\ominus p \oplus p ! \neg p} \models \bot$, i.e., he is going crazy. As a result, the tragedy ends with the suicide of Othello in the final scene.

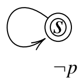


Figure 4: $\mathcal{M}^{\ominus p \oplus p ! \neg p}$.

4 HILBERT-SYSTEM FOR LON

We move on the topic of a proof theory for LON. Hilbert-system for LON (LON), is defined in Table 1. Axioms (4) and (5) indicate what we call positive introspection and negative introspection, respectively. Axiom (**K**&**N**) indicates a relation of knowledge and narrow-mined belief, in which if the agent knows something, he/she also narrow-mindedly believes. This implies that narrow-minded belief is one of the bases of our knowledge, and this view of belief and knowledge can be supported by philosophers and/or psychologists like Hume and Damasio, as discussed in the introduction. Axioms (*RA*$*$) are called *reduction axioms*. Through the reduction axioms and rules, each theorem of LON may be reduced into a theorem of the language $\mathcal{L}_{(\mathbf{KN})}$ which will be shown in Section 5.

We provide some basic definitions and properties for proofs in the next section.

Definition 4.1 (Derivable)**.** *A derivation in* LON *consists of a sequence of formulas of* $\mathcal{L}_{(\mathbf{KN}\ominus\oplus!)}$ *each of which is an instance of an axiom or is the result of applying an inference rule to formula(s) that occur earlier. If* φ *is the last formula in a derivation in* LON*, then* φ *is derivable in* LON*, and we write* $\vdash_{\mathsf{LON}} \varphi$ *(or just* $\vdash \varphi$*).*

Let φ *be a formula of* $\mathcal{L}_{(\mathbf{KN}\ominus\oplus!)}$ *and* Γ *be a subset of languageKakko. Then,* φ is derivable from Γ ($\Gamma \vdash_{\mathsf{LON}} \varphi$) if there exists a finite subset Γ' of Γ such that $\vdash_{\mathsf{LON}} \bigwedge \Gamma' \to \varphi$.

Proposition 4.1. *Let* φ, χ, ψ *be arbitrary formulas of* $\mathcal{L}_{(\mathbf{KN}\ominus\oplus!)}$*. Then the following holds.*

1. $\vdash_{\mathsf{LON}} [*\varphi](\chi \wedge \psi) \leftrightarrow ([*\varphi]\chi \wedge [*\varphi]\psi)$ *(where* $* \in \{\oplus, \ominus, \,!\,\}$*)*
2. $\vdash_{\mathsf{LON}} \varphi \leftrightarrow \chi$ *implies* $\vdash_{\mathsf{LON}} [*\psi]\varphi \leftrightarrow [*\psi]\chi$ *(where* $* \in \{\oplus, \ominus, \,!\,\}$*)*
3. $\vdash_{\mathsf{LON}} \varphi \leftrightarrow \chi$ *implies* $\vdash_{\mathsf{LON}} \Box\varphi \leftrightarrow \Box\chi$ *(where* $\Box \in \{\mathbf{K}, \mathbf{N}\}$

Definition 4.2 (Substitution)**.** *The substitution for formula* $\varphi\binom{p}{\chi}$ *means p appearing in a formula* φ *is*

Table 1: Hilbert-system for LON : LON.

Axioms for knowledge andnarrow−minded belief	
(taut)	all instantiations of propositional tautologies
$(K_{\mathbf{K}})$	$\mathbf{K}(\varphi \to \chi) \to (\mathbf{K}\varphi \to \mathbf{K}\chi)$
$(K_{\mathbf{N}})$	$\mathbf{N}(\varphi \to \chi) \to (\mathbf{N}\varphi \to \mathbf{N}\chi)$
(T)	$\mathbf{K}\varphi \to \varphi$
(4)	$\mathbf{K}\varphi \to \mathbf{K}\mathbf{K}\varphi$
(5)	$\neg\mathbf{K}\varphi \to \mathbf{K}\neg\mathbf{K}\varphi$
(K&N)	$\mathbf{K}\varphi \to \mathbf{N}\varphi$
Inference Rules	
(*MP*)	*From* φ *and* $\varphi \to \chi$*, infer* χ
(*Nec***K**)	*From* φ*, infer* $\mathbf{K}\varphi$
(*Nec***N**)	*From* φ*, infer* $\mathbf{N}\varphi$
(*Nec*$[*]$)	*From* φ*, infer* $[*\chi]\varphi$ where $* \in \{\ominus, \oplus, \,!\,\}$
Reduction Axioms for $[\ominus]$	
(*RA*$\ominus$1)	$[\ominus\psi]p \leftrightarrow p$
(*RA*$\ominus$2)	$[\ominus\psi]\neg\varphi \leftrightarrow \neg[\ominus\psi]\varphi$
(*RA*$\ominus$3)	$[\ominus\psi](\varphi \to \chi) \leftrightarrow ([\ominus\psi]\varphi \to [\ominus\psi]\chi)$
(*RA*$\ominus$4)	$[\ominus\psi]\mathbf{K}\varphi \leftrightarrow \mathbf{K}[\ominus\psi]\varphi$
(*RA*$\ominus$5)	$[\ominus\psi]\mathbf{N}\varphi \leftrightarrow \mathbf{N}[\ominus\psi]\varphi \wedge (\psi \to \mathbf{K}[\ominus\psi]\varphi) \wedge \mathbf{K}(\psi \to [\ominus\psi]\varphi)$
Reduction Axioms for $[\oplus]$	
(*RA*$\oplus$1)	$[\oplus\psi]p \leftrightarrow p$
(*RA*$\oplus$2)	$[\oplus\psi]\neg\varphi \leftrightarrow \neg[\oplus\psi]\varphi$
(*RA*$\oplus$3)	$[\oplus\psi](\varphi \to \chi) \leftrightarrow ([\oplus\psi]\varphi \to [\oplus\psi]\chi)$
(*RA*$\oplus$4)	$[\oplus\psi]\mathbf{K}\varphi \leftrightarrow \mathbf{K}[\oplus\psi]\varphi$
(*RA*$\oplus$5)	$[\oplus\psi]\mathbf{N}\varphi \leftrightarrow \mathbf{N}(\psi \to [\oplus\psi]\varphi)$
Reduction Axioms for $[\,!\,]$	
(*RA*!1)	$[\,!\,\psi]p \leftrightarrow (\psi \to p)$
(*RA*!2)	$[\,!\,\psi]\neg\varphi \leftrightarrow (\psi \to \neg[\,!\,\psi]\varphi)$
(*RA*!3)	$[\,!\,\psi](\varphi \to \chi) \leftrightarrow ([\,!\,\psi]\varphi \to [\,!\,\psi]\chi)$
(*RA*!4)	$[\,!\,\psi]\mathbf{K}\varphi \leftrightarrow (\psi \to \mathbf{K}[\,!\,\psi]\varphi)$
(*RA*!5)	$[\,!\,\psi]\mathbf{N}\varphi \leftrightarrow (\psi \to \mathbf{N}(\psi \to [\,!\,\psi]\varphi))$

replaced by χ*, and defined as follows:*

$$q\binom{p}{\chi} := q \text{ (if } q \neq p),$$
$$q\binom{p}{\chi} := \chi \text{ (if } q = p),$$
$$(\circ\psi)\binom{p}{\chi} := \circ(\psi\binom{p}{\chi}),$$
$$(\psi_1 \to \psi_2)\binom{p}{\chi} := (\psi_1\binom{p}{\chi}) \to (\psi_2\binom{p}{\chi}),$$
$$([*\psi_1]\psi_2)\binom{p}{\chi} := [*\psi_1](\psi_2\binom{p}{\chi}).$$

where $\circ \in \{\neg, \mathbf{K}, \mathbf{N}\}$ *and* $* \in \{\ominus, \oplus, !\}$.

Proposition 4.2. *Let* $* \in \{\ominus, \oplus, !\}$. *If* $\vdash_{\mathsf{LON}} \varphi \leftrightarrow \chi$, *then* $\vdash_{\mathsf{LON}} \psi\binom{p}{\varphi} \leftrightarrow \psi\binom{p}{\chi}$ *for any* $\varphi, \chi, \psi \in \mathcal{L}_{(\mathbf{KN}\ominus\oplus!)}$ *and* $p \in Atom$.

Proof. Fix any $\varphi, \chi \in \mathcal{L}_{(\mathbf{KN}\ominus\oplus!)}$ and $p \in Atom$. Suppose $\vdash_{\mathsf{LON}} \varphi \leftrightarrow \chi$. Then we show $\vdash_{\mathsf{LON}} \psi\binom{p}{\varphi} \leftrightarrow \psi\binom{p}{\chi}$ By induction on $\psi \in \mathcal{L}_{(\mathbf{KN}\ominus\oplus!)}$. □

5 COMPLETENESS

Let us move onto a proof of the completeness theorem of LON with a similar argument in (van Ditmarsch et al., 2008, Section 5).

5.1 Semantic Completeness of LON'

Let *the language* $\mathcal{L}_{(\mathbf{KN})}$ be our formal language $\mathcal{L}_{(\mathbf{KN}\ominus\oplus!)}$ without announcement operators ($[\ominus], [\oplus]$ and $[\,!\,]$). For an en-model $\mathcal{M}$ and $s \in \mathcal{D}(\mathcal{M})$ and $\varphi \in \mathcal{L}_{(\mathbf{KN})}$, the satisfaction relation $\mathcal{M}, s \models \varphi$ is naturally defined by following the definition in Section 2. Additionally, *Hilbert-system* LON' is also generated by rejecting the reduction axioms and inference rules of $(Nec[\ominus]), (Nec\oplus)$ and $(Nec[\,!\,])$ in Table 1. Note that definitions of the derivation and derivability of LON' are given in the same manner as that of LON in Definition 4.1.

Theorem 5.1 (Soundness of LON')**.** *For any formula* $\varphi \in \mathcal{L}_{(\mathbf{KN})}$,

$$\vdash_{\mathsf{LON}'} \varphi \textit{ implies } \models \varphi.$$

Proof. Fix any $\varphi \in \mathcal{L}_{(\mathbf{KN})}$ such that φ is derivable in LON'. We show that φ is valid by induction on the height of the derivation. In the base case, the derivation height is 0 i.e., it consists of only an axiom. Therefore, we show the validity of each axiom of LON. □

A direct proof of the completeness theorem of LON' can be shown in a usual manner with Lindenbaum's lemma.

Theorem 5.2 (Completeness for LON' w.r.t. the semantics of $\mathcal{L}_{(\mathbf{KN})}$)**.** *For any* $\varphi \in \mathcal{L}_{(\mathbf{KN})}$*, the following holds:*

$$\models \varphi \textit{ implies } \vdash_{\mathsf{LON}'} \varphi.$$

5.2 Semantic Completeness of LON

Based on the completeness theorem of LON', we expand the discussion to the completeness of LON. A proof of the completeness theorem of LON is given in this section by the reduction method whose basic idea was introduced in the previous work (Plaza, 1989, Theorem 2.7). The essential idea of this method is based on the fact that every formula in $\mathcal{L}_{(\mathbf{KN}\ominus\oplus!)}$ is reducible into a formula in $\mathcal{L}_{(\mathbf{KN})}$ which will be shown in Lemma 5.3.

Remark 5.1. We note that reduction axioms for sequential announcement operators e.g.,

$$(RA!6) \quad [\,!\,\chi][\,!\,\psi]\varphi \leftrightarrow [\,!\,(\chi \wedge [\,!\,\chi]\psi)]\varphi$$

are not included since, without them, any formula with announcement operators can be reducible. It is known that there are at least two strategies to reduce a formula with announcement operators into a formula without any such operator. Let us consider the formula $[!p][!q]r - (i)$. One approach, we may call it 'outermost strategy', focuses on the outermost occurrence of announcement operator, for example $[!p]$ of the above formula (i). Following this strategy, an axiom like $(RA!6)$ is required for reducing the formula. By using $(RA!6)$, we may obtain $[!(p \wedge [!p]q)]q$. Then $(RA!1)$ becomes applicable, and so we obtain the formula which does not include any announcement operator but is equivalent to the initial formula. This approach is introduced by (van Ditmarsch et al., 2008). The other strategy may be called 'innermost strategy' and focuses on the innermost occurrence of announcement operator, for example $[!q]$ of (i). Thus, by applying $(RA!1)$ to the innermost occurrence i.e., $[!q]r$, we obtain $[!p](q \to r)$. After that, $(RA!3)$ and $(RA!1)$ are subsequently applicable, and so we obtain the formula without any announcement operator but equivalent to the initial formula of (i). The latter strategy does not require reduction axioms for reducing sequential announcement operators into a single. Therefore, we employ this strategy to avoid introducing many and messy axioms.[1] The idea of this innermost strategy was introduced by (van Benthem et al., 2006), and (van Benthem, 2011, p.54). Furthermore, an attentive proof for reducibility of a formula of Dynamic logic into a formula of standard modal logic

[1]If we follow the outermost strategy, six additional axioms (e.g, axioms for reducing combination of $[\oplus A][\ominus B]$ and $[\,!\,A][\oplus B]$ etc.) are required.

by using the innermost strategy is given in (Aucher, 2003, Proposition 3.3.5).[2]

At first, we treat the soundness theorem of it which is straightforward.

Theorem 5.3 (Soundness of LON). *For any formula* $\varphi \in \mathcal{L}_{(\mathbf{KN}\ominus\oplus!)}$,

$$\vdash_{\mathsf{LON}} \varphi \textit{ implies } \models \varphi.$$

Proof. For the soundness theorem of LON, it suffices to show the validity of reduction axioms, additional axioms to that of LON′ and additional inference rules $(Nec[\ominus]), (Nec[\oplus])$ and $(Nec[\,!\,])$. The validity of additional rules and axioms are also easily shown by following semantics of LON. □

Next, we give some definitions and lemmas for proof of the completeness.

Definition 5.1 (Length). *The length function* $\ell : \mathcal{L}_{(\mathbf{KN}\ominus\oplus!)} \to \mathbb{N}$ *is inductively defined as follows:*

$\ell(p) := 1,$	$\ell(\mathbf{N}\varphi) := 1+\ell(\varphi),$
$\ell(\neg\varphi) := 1+\ell(\varphi),$	$\ell([\oplus\varphi]\chi) := (5+\ell(\varphi))^{\ell(\chi)},$
$\ell(\varphi\to\chi) := 1+\ell(\varphi)+\ell(\chi),$	$\ell([\ominus\varphi]\chi) := (5+\ell(\varphi))^{\ell(\chi)},$
$\ell(\mathbf{K}\varphi) := 1+\ell(\varphi),$	$\ell([\,!\,\varphi]\chi) := (5+\ell(\varphi))^{\ell(\chi)}.$

With these settings, we may show the following lemma.

Lemma 5.1. *Let* $*$ *be* $\oplus, \ominus$ *or* $!$. *Then for all reduction axioms* $[*\varphi]\chi \leftrightarrow \psi$, $\ell([*\varphi]\chi) > \ell(\psi)$ *holds.*

Proof. We only confirm the following case.

Case of $(RA\ominus 5)$**.** The less-than relation $\ell(\mathbf{N}[\ominus\psi]\varphi \wedge (\psi \to \mathbf{K}[\ominus\psi]\varphi) \wedge \mathbf{K}(\psi \to [\ominus\psi]\varphi)) < \ell([\ominus\chi]\mathbf{N}\varphi)$ holds by the following equations:

$$\begin{aligned}
&\ell(\mathbf{N}[\ominus\psi]\varphi \wedge (\psi \to \mathbf{K}[\ominus\psi]\varphi) \wedge \mathbf{K}(\psi \to [\ominus\psi]\varphi))\\
&= \ell(\mathbf{N}[\ominus\psi]\varphi \to \neg(\psi \to \mathbf{K}[\ominus\psi]\varphi) \wedge \mathbf{K}(\psi \to [\ominus\psi]\varphi))\\
&= 2+\ell(\mathbf{N}[\ominus\psi]\varphi)+\ell(\varphi \to \mathbf{K}[\ominus\psi]\varphi)+\ell(\mathbf{K}\varphi \to \mathbf{K}[\ominus\psi]\varphi)\\
&= 8+\ell([\ominus\psi]\varphi)+\ell(\varphi)+\ell([\ominus\psi]\varphi)+\ell(\varphi)+\ell([\ominus\psi]\varphi)\\
&= 8+2\cdot\ell(\varphi)+3\cdot(5+\ell(\psi))^{\ell(\varphi)}
\end{aligned}$$

Then we can prove that $\ell([\ominus\psi]\mathbf{N}\varphi) = k^{1+\ell(\varphi)} > 8+2\cdot\ell(\varphi)+3\cdot k^{\ell(\varphi)}$ (where $k = (5+\ell(\psi)) \geq 6$) holds.[3]

□

[2] We add one more comment for a technical difference between the two strategies. In the outermost strategy of public announcement logic, we need to include axiom like $(RA!6)$ to reduce sequential announcement operators into a single, but the inference rule of $(Nec[!])$ is derivable. On the other hand, the rule is indispensable in the case of the innermost strategy, instead of economizing the number of axioms.

[3] Let $\ell(\varphi) = n$ and $f_n(k) \equiv k^{n+1} - 3k^n - 2n - 8$. Then obviously $f_n(k) = k^n(k-3) - 2n - 8 > 0$ for $k \geq 6$ for any fixed $n \geq 1$, as well as for fixed $n \geq 1$ for any $k \geq 6$.

Lemma 5.2. *For any* $p \in Atom$, $\varphi, \chi, \psi \in \mathcal{L}_{(\mathbf{KN}\ominus\oplus!)}$, *if* $\ell(\varphi) > \ell(\chi)$ *and* $\psi\binom{p}{\varphi} \neq \psi\binom{p}{\chi}$, *then* $\ell(\psi\binom{p}{\varphi}) > \ell(\psi\binom{p}{\chi})$ *holds.*

Proof. Fix any $p \in Atom$ and $\varphi, \chi \in \mathcal{L}_{(\mathbf{KN}\ominus\oplus!)}$. Then assume $\ell(\varphi) > \ell(\chi)$. We conduct the proof by induction on $\psi \in \mathcal{L}_{(\mathbf{KN}\ominus\oplus!)}$.

Case of ψ **is of the form** $\psi_1 \to \psi_2$**.** Assume $(\psi_1 \to \psi_2)\binom{p}{\varphi} \neq (\psi_1 \to \psi_2)\binom{p}{\chi}$. We show $\ell((\psi_1 \to \psi_2)\binom{p}{\varphi}) > \ell((\psi_1 \to \psi_2)\binom{p}{\chi})$. Therefore, it suffices to show $\ell(\psi_1\binom{p}{\varphi}) + \ell(\psi_2\binom{p}{\varphi}) > \ell(\psi_1\binom{p}{\chi}) + \ell(\psi_2\binom{p}{\chi})$. From the assumption, we obtain that at least one of C_1 and C_2 satisfies $\psi_i\binom{p}{\varphi} \neq \psi_i\binom{p}{\chi}$. Without loss of generality, assume that $\psi_1\binom{p}{\varphi} \neq \psi_1\binom{p}{\chi}$. By induction hypothesis, we obtain $\ell(\psi_1\binom{p}{\varphi}) > \ell(\psi_1\binom{p}{\chi})$. Therefore, no matter whether $\psi_2\binom{p}{\varphi} \neq \psi_2\binom{p}{\chi}$ or not, we obtain the goal.

Other cases are similar to the above. □

Definition 5.2. $\ell' : \mathcal{L}_{(\mathbf{KN}\ominus\oplus!)} \to \mathbb{N}$ *is defined as follows.*

$$\ell'(\varphi) := \begin{cases} 0 & \textit{if } \varphi \in \mathcal{L}_{(\mathbf{KN})} \\ \ell(\varphi) & \textit{otherwise} \end{cases}$$

Lemma 5.3 (Reduction lemma). *For any* $\varphi \in \mathcal{L}_{(\mathbf{KN}\ominus\oplus!)}$, *there exists* $\psi \in \mathcal{L}_{(\mathbf{KN})}$ *such that* $\vdash_{\mathsf{LON}} \varphi \leftrightarrow \psi$.

Proof. By induction on $\ell'(\varphi)$. We only treat the following case.

Case: $\ell'(\varphi) > 0$**.** In this case, $\varphi \in \mathcal{L}_{(\mathbf{KN}\ominus\oplus!)}$ includes at least one subformula which is of the form $[*\chi_1]\chi_2$ (where $* \in \{\oplus, \ominus, !\}$ and $\chi_1 \in \mathcal{L}_{(\mathbf{KN}\ominus\oplus!)}, \chi_2 \in \mathcal{L}_{(\mathbf{KN})}$). On the other hand, there is a reduction axiom which has the form of $[*\chi_1]\chi_2 \leftrightarrow \chi_3$, and let this reduction axiom be $(RA*)$. The formula φ is equal to $\mathcal{D}\binom{p}{[*\chi_1]\chi_2}$ and φ' is equal to $\mathcal{D}\binom{p}{\chi_3}$ for some $\mathcal{D} \in \mathcal{L}_{(\mathbf{KN}\ominus\oplus!)}$, and so fix such $\mathcal{D}$. Then we may obtain the following derivation.

1. $\vdash [*\chi_1]\chi_2 \leftrightarrow \chi_3$ (RA*)
2. $\vdash \varphi \leftrightarrow \varphi'$ 1 and Proposition 4.2
3. $\vdash \varphi' \leftrightarrow \psi$ Induction hypothesis
4. $\vdash (\varphi \leftrightarrow \varphi') \to ((\varphi' \leftrightarrow \psi) \to (\varphi \leftrightarrow \psi))$ (taut)
5. $\vdash \varphi \leftrightarrow \psi$ 2,3 and 4 with (MP)

where $\varphi = \mathcal{D}\binom{p}{[*\chi_1]\chi_2}$ and $\varphi' = \mathcal{D}\binom{p}{\chi_3}$. Induction hypothesis in the above derivation is applicable, since the less-than relation $\ell'(\mathcal{D}\binom{p}{[*\chi_1]\chi_2}) > \ell'(\mathcal{D}\binom{p}{\chi_3})$ holds by Lemma 5.1 and Lemma 5.2.

□

Actually, Lemma 5.3 is the core of the proof of the completeness theorem. Through this, we may easily show the theorem as follows.

Theorem 5.4 (Completeness of LON w.r.t. the semantics of $\mathcal{L}_{(\mathbf{KN}\ominus\oplus!)}$)**.** *For any formula* $\varphi \in \mathcal{L}_{(\mathbf{KN}\ominus\oplus!)}$*, the following holds:*

$$\models \varphi \textit{ implies } \vdash_{\mathsf{LON}} \varphi.$$

Proof. Fix any $\varphi \in \mathcal{L}_{(\mathbf{KN}\ominus\oplus!)}$ such that $\models \varphi$. By Lemma 5.3, we obtain $\vdash_{\mathsf{LON}} \varphi \leftrightarrow \chi$ for some $\chi \in \mathcal{L}_{(\mathbf{KN})}$. Then consider such $\chi \in \mathcal{L}_{(\mathbf{KN})}$. By Theorem 5.3 (the soundness of LON), we obtain $\models \varphi \leftrightarrow \chi$. With the assumption $\models \varphi$, we have $\models \chi$. Next, by Theorem 5.2 (the completeness of LON'), we obtain $\vdash_{\mathsf{LON}'}\chi$, and so $\vdash_{\mathsf{LON}}\chi$ trivially holds; therefore, we obtain $\vdash_{\mathsf{LON}} \varphi$ with $\vdash_{\mathsf{LON}} \varphi \leftrightarrow \chi$ again. That is what we desired. □

6 RELATED WORKS

In this section, we introduce some related epistemic/doxastic logics. An epistemic logic for implicit and explicit belief by (Velázquez-Quesada, 2014) is perhaps the closest concept we can find to that of LON. This logic is based on the logic of awareness logic (van Benthem and Velázquez-Quesada, 2010), and it distinguishes the sense of belief into two, implicit and explicit belief, to avoid the logical omniscience in epistemic logic. A traditional approach to mix knowledge and belief operators, sometimes called epistemic-doxastic logic (e.g., see (Voorbraak, 1993)), is another system similar to ours since **K** and **N** of LON may be interpreted as a mixture of these two different human tendencies.

One of differences between LON and the above existing works may relate to the definition of the satisfaction relation of LON:

$$\mathcal{M}, s \models [\ominus\varphi]\chi \quad \textit{iff} \quad \mathcal{M}^{\ominus\varphi}, s \models \chi,$$
$$\text{where } Q^{\ominus\varphi} := Q \cup \{(s,t) \in R \mid s \in [\![\varphi]\!]_{\mathcal{M}} \text{ or } t \in [\![\varphi]\!]_{\mathcal{M}}\}.$$

Here, we include a mechanism of adding arrows i.e., a mechanism in which some of the information may confuse the agent.

In addition, there are some other attempts to introduce a distinction in our belief/knowledge from a different point of view. Intuitionistic epistemic logic (Artemov and Protopopescu, 2014; Williamson, 1992) is one of them; this epistemic logic is based on intuitionistic logic, which distinguishes knowledge into two: standard knowledge, which normal epistemic logics treat and knowledge in the strict sense. In other words, this aims at introducing a distinction in knowledge, more strict and rational knowledge and not strict knowledge, which is an opposite perspective to our attempt, which introduced a distinction between belief with passion and knowledge.

There are also some logics which deal with human emotion; for example (Lorini and Schwarzentruber, 2011) and (Dastani and Lorini, 2012). We may, for the further development, need to consider relevance to these existing logics about emotion.

7 CONCLUSION AND FURTHER DIRECTIONS

We introduced logic of narrow-minded belief (LON), a variant of dynamic epistemic logic. This aims to formally express a human's passionate and narrow-minded belief, and as an example of the application of LON, we formalized Shakespeare's play *Othello*. Philosophers and neuropsychologists believe that passion, or belief affected by passion, is an indispensable factor and even a basis for our reason. Without passion or emotions, human intelligence may be never realized. Therefore, we hope that our attempt in the present work will contribute to formal expressions of the human mind.

It may be possible to further develop our attempt in various directions. For example, we did not regard the problem of the logical omniscience; the logic of awareness is one of the candidates to be added to LON, as it is difficult to interpret the meaning of awareness in the context of passion. Another interesting feature that should be considered and added to LON is 'a lie' as it pertains to dynamic epistemic logic by van Ditmarsch (van Ditmarsch, 2011). Actually, Iago's rumor should be regarded as a lie, as our passion or narrow-mindedness is easily affected by such dubious information. Therefore, it might be interesting to consider these aspects in future researches regarding the logic of passion.

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