On Dynamic Output Feedback H_∞ Control for Positive Discrete-time Delay Systems

Baozhu Du
School of Automation, Nanjing University of Science and Technology, 210094, P. R. China

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Abstract: This paper is devoted to the H_∞ control design of positive discrete-time systems with multiple delays. Novel bounded real lemma is presented first via linear matrix inequality technique, which reveals that H_∞ norms of a discrete-time positive system with time delays both in dynamic and output equations are identical to that of the corresponding delay-free system. Necessary and sufficient conditions for positivity preserving H_∞ stabilization via a dynamic output feedback control are established in the forms of matrix equalities, that guaranteeing the closed-loop system not only to be asymptotically stable and positive, but also to have a desired H_∞ performance. The proposed results are extended to interval uncertain positive systems with time delay. Finally, an example is given to illustrate the effectiveness of the obtained design scheme.

1 INTRODUCTION

Positive systems are a class of systems whose state variables are never negative, for any given nonnegative initial state and nonnegative input. Lots of stability and stabilization problems for time-delayed positive systems have been reported in the literature, see, for instance (Gao et al., 2004b), (Cui et al., 2018). Necessary and sufficient conditions based on linear programming technique were given to guarantee the asymptotical stability of discrete-time positive systems with constant delays in (Liu, 2009), which proved that the magnitudes of delays have no impact on system stability. Stability analysis of positive systems with bounded time-varying delays was studied in (Liu et al., 2010). Exponential stability of positive time-delayed systems was investigated in (Zhu et al., 2012) by the Lyapunov-Krasovskii functional based method, and diagonal Riccati stability criteria was presented in (Mason, 2012) by using the separating hyperplane theorem.

For the controller synthesis, an output feedback controller has to be used if no full access to the system states (Wang et al., 2015), (Shu et al., 2012), (Zhang et al., 2018). There are generally two types of strategies to avoid NP-hard problem (Blondel and Tsitsiklis, 1995). One is so-called relaxation, which is easy to implement, but conservatism may be introduced in some cases (Gao et al., 2004a). The other strategy is the local optimization which minimizes the objective function near the feasible point. Most accurate methods to static output feedback synthesis involve local optimization, for instance, the direct iterative procedure (D-K iteration), iterative linear matrix inequality (ILMI) and the cone complementarity linearization (CCL) (Geromel et al., 1994). The free-weighting matrix method proposed in (He et al., 2007) has reduced the conservatism in controller synthesis, but always introduces extra coupling terms among controller gain, Lyapunov matrices and system matrices (Mirkin and Gutman, 2005). To decouple these cross-product terms, an augmentation approach was proposed in (Shu and Lam, 2009) provided an equivalent form of the H_∞ stabilization criterion for positive delay-free systems.

The bounded real lemma (BRL) and Kalman-Yakubovich-Popov(KYP) lemma for linear positive systems without time delays was presented by T. Tanaka and C. Langbort in (Tanaka and Langbort, 2010) , in which the KYP lemma made the condition of H_∞ controller design be convex and tractable with the help of the small gain theorem and the hyperplane separation theorem. Strict/non-strict inequality versions of KYP lemma for single-input single-output discrete-time positive systems without time delays were developed in (Najson, 2013) where a quadratic Lyapunov function was formulated by a diagonal Lyapunov matrix (Farina and Rinaldi, 2000). BRL in terms of matrix inequality for continuous-time positive systems with time delays in states was

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provided in (Zhang et al., 2015), and the criteria for dynamic output feedback H_∞ stabilizability were also proposed. To the authors' knowledge, there are still no related results to the H_∞ stabilization problem for discrete-time positive system with discrete delays.

Based on the above observations, this paper is motivated to present BRL of discrete-time positive systems with time delays for the first time, with further discussion on H_∞ control by means of dynamic output feedback control strategy. The remaining parts of this paper are organized as follows. Preliminary is introduced in Section 2 and a novel BRL is established in Section 3. Necessary and sufficient conditions are proposed to prove that H_∞ performance of positive time-delayed systems are independent on the magnitudes of delays. In the aspect of controller synthesis, necessary and sufficient conditions are presented in Section 4 to design dynamic output feedback controllers, which leads to the closed-loop system to possess asymptotical stability, positivity, and desired H_∞ performance simultaneously. Section 5 extends results to interval linear discrete-time systems with both time delays and uncertainties. A numerical example is given to illustrate the effectiveness of the obtained results in Section 6.

2 PRELIMINARIES

The notations throughout this paper are fairly standard. For two matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{m\times n}$, $A \geq B$, $A > B$, $A \gg (\ll) B$, respectively, denote that $A_{ij} \geq B_{ij}$, $A_{ij} \geq B_{ij}$ but $A \neq B$, $A_{ij} > (<) B_{ij}$, for all $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$. For $A \in \mathbb{R}^{n\times n}$, $A \succeq 0$ and $A \prec 0$ mean that A is a positive semidefinite and a negative definite matrix, respectively. $\langle X,Y\rangle = \mathrm{trace}(XY)$ is the inner product on $\mathbb{S}^n$. The asterisk "$*$" in a matrix represents a term which can be induced by symmetry. Moreover,

$N_+, \mathbb{R}$	set of positive integers, set of real numbers
$\mathbb{R}^n$	set of n-dimensional real vectors
$\mathbb{R}^{m\times n}$	set of $m\times n$ real matrieces
$\bar{\mathbb{R}}^n_+, \mathbb{R}^n_+$	nonnegative and positive orthants of $\mathbb{R}^n$
$\mathbb{S}^n$	space of n-th order real symmetric matrices
$\mathbb{D}^{n\times n}_+$	set of all diagonal positive definite matrices
e_i	vector with 1 in ith position and 0 elsewhere
$\mathbf{1}, I$	vector $[1,1,\ldots,1]^T$, identity matrix
A_{ij}	ith component of matrix A
A^T	transpose of A
$\mathrm{trace}(A)$	trace of matrix A
$\rho(A)$	spectral radius of matrix A
$\bar{\sigma}(A)$	maximum singular value of matrix A
$\mathfrak{D}(A)$	vector composed of diagonal entries of A

Consider a linear discrete-time positive system with time delays in state and output equations as follows,

$$\begin{aligned}\Sigma_0:\ x(k+1) &= Ax(k) + \sum_{i=1}^{q} A_i x(k-d_i) + B\omega(k),\\ z(k) &= Cx(k) + \sum_{i=1}^{q} C_i x(k-d_i) + D\omega(k),\\ x(k) &= \phi(k),\ k\in[-d,0],\end{aligned} \tag{1}$$

where $x(k) \in \bar{\mathbb{R}}^n_+$, $\omega(k) \in \bar{\mathbb{R}}^m_+$, $z(k) \in \bar{\mathbb{R}}^p_+$ are the state, exogenous input and output vectors, respectively. A, A_i, B, C, C_i, D are known real matrices with appropriate dimensions, d_i is a constant time delay, $\phi(k) \in \bar{\mathbb{R}}^n_+$ is the vector-valued initial function on $[-d,0]$ with $d \triangleq \max\{d_i\}$, $i = 1,2,\ldots,q$. Some necessary definitions and lemmas are provided first, which are useful in the subsequent technical development for linear time-delay positive systems.

Definition 1. *Matrix $A \in \mathbb{R}^{n\times n}$ is Schur stable if $\rho(A) < 1$.*

Lemma 1. *((Berman and Plemmons, 1979)) For two matrices $A \in \mathbb{R}^{n\times n}_+$, $B \in \mathbb{R}^{n\times n}_+$, $\rho(A) \geq \rho(B)$ if $A \geq B$.*

Lemma 2. *(Liu, 2009) For positive system Σ_0, the following statements hold:*

(i) System Σ_0 is positive if and only if $A \geq 0$, $B \geq 0$, $C \geq 0$, $D \geq 0$, $A_i \geq 0$, $C_i \geq 0$, $i = 1,2,\ldots,q$;

(ii) System Σ_0 is asymptotically stable if and only if $\rho\left(A + \sum_{i=1}^{q} A_i\right) < 1$.

The transfer function matrix of system Σ_0 is given by

$$G_0(z) = \left(C + \sum_{i=1}^{q} z^{-d_i} C_i\right)\left(zI - A - \sum_{i=1}^{q} z^{-d_i} A_i\right)^{-1} B + D,$$

and its H_∞ norm is defined as

$$||G_0||_\infty = sup_{\theta\in[0,2\pi)}\ \bar{\sigma}(G(e^{j\theta})).$$

A sufficient condition to check H_∞ characteristics of system Σ_0 with $q = 1$ has been established in (Gao et al., 2004a) as follows.

Lemma 3. *(Gao et al., 2004a) Positive system Σ_0 with $q = 1$ is asymptotically stable and $||G_0||_\infty < \gamma$ if there exist matrices $P \succ 0$ and $Q \succ 0$ such that*

$$M + \mathrm{diag}\{Q, -Q, 0\} \prec 0, \tag{2}$$

where

$$M = \begin{bmatrix} A^TPA - P + C^TC & A^TPA_1 + C^TC_1 & C^TD + A^TPB \\ C_1^TC + A_1^TPA & A_1^TPA_1 + C_1^TC_1 & A_1^TPB + C_1^TD \\ D^TC + B^TPA & B^TPA_1 + D^TC_1 & B^TPB + D^TD - \gamma^2 I \end{bmatrix}.$$

3 BOUNDED REAL LEMMA (BRL)

In this section, we shall point out that H_∞ performance of the discrete-time positive linear system Σ_0 with constant delays is insensitive to the magnitude of the delays. Our purpose is to give a characterization on the BRL for system Σ_0 with multiple time delays. To this end, we first introduce two nominal delay-free positive systems:

$$\begin{aligned}\Sigma_1: \; x(k+1) &= \tilde{A}x(k) + B\omega(k),\\ z(k) &= \tilde{C}x(k) + D\omega(k),\end{aligned} \tag{3}$$

$$\begin{aligned}\Sigma_2: \; x(k+1) &= Ax(k) + B\omega(k),\\ z(k) &= Cx(k) + D\omega(k).\end{aligned} \tag{4}$$

For simplicity, define $\tilde{A} = A + \tilde{A}_d$, $\tilde{A}_d = \sum_{i=1}^{q} A_i$, $\tilde{C} = C + \tilde{C}_d$, $\tilde{C}_d = \sum_{i=1}^{q} C_i$. The transfer functions of systems Σ_1 and Σ_2 are, respectively, given by

$$G_1(z) = \tilde{C}\left(zI - \tilde{A}\right)^{-1}B + D$$
$$G_2(z) = C(zI - A)^{-1}B + D$$

with $z = e^{j\theta}$, $\theta \in [0, 2\pi)$. It has been pointed out in (Najson, 2013) that, if system Σ_2 is positive and asymptotically stable, $||G_2||_\infty = \bar{\sigma}(G_2(1))$. On the basis of this fact, the following lemma can be obtained which is useful sequentially.

Lemma 4. *If system Σ_0 is positive, asymptotically stable and $||G_0||_\infty < \gamma$, then $||G_2||_\infty \le ||G_1||_\infty < \gamma$.*

Proof: If positive system Σ_0 is asymptotically stable and $||G_0||_\infty < \gamma$, one has $\bar{\sigma}(G_0(1)) < \gamma$. Obviously, $||G_1||_\infty = \bar{\sigma}(G_1(1)) = \bar{\sigma}(G_0(1)) < \gamma$. It follows from Lemma 2 that

$$(I - \tilde{A})^{-1} = \sum_{k=0}^{\infty} \tilde{A}^k \ge \sum_{k=0}^{\infty} A^k = (I - A)^{-1} \ge 0,$$

which leads to $0 \le G_2(1) \le G_1(1)$. According to Lemma 1, $||G_2||_\infty = \bar{\sigma}(G_2(1)) \le \bar{\sigma}(G_1(1)) = ||G_1||_\infty < \gamma$ is derived. □

Theorem 1. *[**Single delay**] When $q = 1$. System Σ_0 is asymptotically stable and $||G_0||_\infty < \gamma$ if and only if there exist $P \in \mathbb{D}_+^{n\times n}$ and $Q \in \mathbb{D}_+^{n\times n}$ such that inequality* (2) *holds.*

Proof: Necessity. Define a discrete-time system

$$\begin{aligned}\hat{\Sigma}_0: \; x(k+1) &= Ax(k) + A_1x(k-d_1) + B\omega(k),\\ z(k) &= \hat{C}x(k) + \hat{C}_1x(k-d_1) + \hat{D}\omega(k),\\ x(k) &= \phi(k), \; k \in [-d_1, 0],\end{aligned} \tag{5}$$

in which $\hat{C} = \frac{1}{\gamma}C$, $\hat{C}_1 = \frac{1}{\gamma}C_1$, $\hat{D} = \frac{1}{\gamma}D$. Its transfer function matrix is denoted by $\hat{G}_0(z)$. If system Σ_0 with $q = 1$ is positive, asymptotically stable and $||G_0||_\infty < \gamma$, one has $||\hat{G}_0||_\infty = ||\frac{1}{\gamma}G_0||_\infty < 1$. At this point, it turn s to prove that, if system $\hat{\Sigma}_0$ is positive, asymptotically stable and $||\hat{G}_0||_\infty < 1$, there must exist $P \in \mathbb{D}_+^{n\times n}$ and $Q \in \mathbb{D}_+^{n\times n}$ satisfying

$$\hat{M} + \mathrm{diag}\{Q, -Q, 0\} < 0, \tag{6}$$

where $\hat{M}$ is M defined in Lemma 3 with C, C_1 and D, respectively, replaced by $\hat{C}$, $\hat{C}_1$ and $\hat{D}$, and $\gamma = 1$.

The proof will be given by contradiction. Suppose that, for every $P \in \mathbb{D}_+^{n\times n}$, there does not exist any nonzero $Q \in \mathbb{D}_+^{n\times n}$ such that LMI (6) holds. Define two sets

$$S_1 \triangleq \left\{ \hat{M} + \mathrm{diag}\{Q, -Q, 0\} \,\middle|\, Q \in \mathbb{D}_+^{n\times n} \right\},$$
$$S_2 \triangleq \{R | R \prec 0, R \in \mathbb{S}^{2n+m}\}.$$

It can be easily verified that both sets S_1 and S_2 are convex and nonempty. The intersection of sets S_1 and S_2 is empty means that $S_1 \cap S_2 = \emptyset$. It follows from the separating hyperplane theorem (Boyd and Balakrishnan, 2004) and convex analysis (see (Rockafellar, 2015)) that two disjoint convex sets can be separated by a hyperplane, that is, there exists a nonzero matrix $H \in \mathbb{S}^{2n+m}$ such that

$$\langle H, Y \rangle \ge 0, \; \forall Y \in S_1, \tag{7}$$
$$\langle H, X \rangle < 0, \; \forall X \in S_2. \tag{8}$$

From condition (8), that is, $\langle H, X \rangle = \mathrm{trace}(HX) < 0$, one can easily verify that $H \succeq 0$. Defining

$$H = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{12}^T & H_{22} & H_{23} \\ H_{13}^T & H_{23}^T & H_{33} \end{bmatrix} \succeq 0,$$

inequality (7) yields that, for any $Q \in \mathbb{D}_+^{n\times n}$,

$$\begin{aligned}&\mathrm{trace}\left(H(\hat{M} + \mathrm{diag}\{Q, -Q, 0\})\right)\\ &= \mathrm{trace}(H\hat{M}) + \mathrm{trace}\left((H_{11} - H_{22})Q\right) \ge 0.\end{aligned}$$

It follows from $Q > 0$ that

$$\mathrm{trace}(H\hat{M}) \ge 0, \tag{9}$$
$$\mathfrak{D}(H_{11} - H_{22}) \ge 0, \tag{10}$$

otherwise, there must exist nonzero $H \succeq 0$ and $Q \in \mathbb{D}_+^{n\times n}$ such that $\mathrm{trace}(H\hat{M}) + \mathrm{trace}\left((H_{11} - H_{22})Q\right) < 0$ for any fixed $P \in \mathbb{D}_+^{n\times n}$.

Define a nonzero vector $h \triangleq \begin{bmatrix} h_1^T & h_2^T & h_3^T \end{bmatrix}^T \in \mathbb{R}_+^{2n+m}$ with $h_i = \sqrt{H_{ii}}$, $i = 1, 2, 3$. From inequality (9) and Lemma 1, one has $\mathrm{trace}(hh^T\hat{M}) \ge 0$ which is equivalent to,

$$\begin{aligned}\Omega \triangleq & h_1^T(A^TPA-P+\hat{C}^T\hat{C})h_1+h_2^T(A_1^TPA+\hat{C}_1^T\hat{C})h_1\\ &+h_3^T(B^TPA+\hat{D}^T\hat{C})h_1+h_1^T(A^TPA_1+\hat{C}^T\hat{C}_1)h_2\\ &+h_2^T(A_1^TPA_1+\hat{C}_1^T\hat{C}_1)h_2+h_3^T(B^TPA_1+\hat{D}^T\hat{C}_1)h_2\\ &+h_1^T(A^TPB+\hat{C}^T\hat{D})h_3+h_2^T(A_1^TPB+\hat{C}_1^T\hat{D})h_3\\ &+h_3^T(B^TPB+\hat{D}^T\hat{D}-I)h_3\geq 0.\end{aligned} \tag{11}$$

From (10), one has $h_1 \geq h_2 \geq 0$. Note that vector h is nonzero, and two cases will be discussed as follows.

Case 1: $h_2 = 0$. Inequality (11) leads to

$$\begin{aligned}&h_1^T(A^TPA-P+\hat{C}^T\hat{C})h_1+h_3^T(B^TPA+\hat{D}^T\hat{C})h_1\\ &+h_1^T(A^TPB+\hat{C}^T\hat{D})h_3+h_3^T(B^TPB+\hat{D}^T\hat{D}-I)h_3\geq 0\end{aligned} \tag{12}$$

which can be rewritten as

$$\begin{bmatrix} h_1 \\ h_3 \end{bmatrix}^T \Upsilon \begin{bmatrix} h_1 \\ h_3 \end{bmatrix} \geq 0.$$

in which

$$\Upsilon \triangleq \begin{bmatrix} A^TPA-P+\hat{C}^T\hat{C} & A^TPB+\hat{C}^T\hat{D} \\ B^TPA+\hat{D}^T\hat{C} & B^TPB+\hat{D}^T\hat{D}-I \end{bmatrix}.$$

It means that there exists a nonzero vector $\begin{bmatrix} h_1^T & h_3^T \end{bmatrix}^T$ such that the above inequality holds. Equivalently, there does not exist $P \in \mathbb{D}_+^{n\times n}$ such that $\Upsilon \prec 0$. According to the KYP Lemma (in (Rantzer, 2016)) and Lemma 4, one has $||\frac{1}{\gamma}G_2||_\infty \geq 1$. This is a contradiction.

Case 2: $h_2 > 0$. Due to the fact that $h_1 \geq h_2$, Ω defined in condition (11) satisfies

$$\Omega \leq \begin{bmatrix} h_1 \\ h_3 \end{bmatrix}^T \begin{bmatrix} (A+A_1)^TP(A+A_1)-P+(\hat{C}+\hat{C}_1)^T(\hat{C}+\hat{C}_1) & (A+A_1)^TPB+(\hat{C}+\hat{C}_1)^T\hat{D} \\ B^TP(A+A_1)+\hat{D}^T(\hat{C}+\hat{C}_1) & B^TPB+\hat{D}^T\hat{D}-I \end{bmatrix} \begin{bmatrix} h_1 \\ h_3 \end{bmatrix},$$

which is inconsistent with the fact that $||\frac{1}{\gamma}G_1||_\infty < 1$.

Hence, if system Σ_0 is positive, asymptotically stable and $\|G_0\|_\infty < \gamma$, there must exist $P \in \mathbb{D}_+^{n\times n}$ and $Q \in \mathbb{D}_+^{n\times n}$ satisfying $M + \mathrm{diag}\{Q, -Q, 0\} \prec 0$.

Sufficiency condition can be immediately obtained from Lemma 3 and Theorem 1 given in (Wu et al., 2009). This completes the proof. □

After an algebraic manipulation, a simple equivalent form of (2) in Theorem 1 can be obtained in the following corollary, in which the matrix variable Q appearing in Theorem 1 has been removed.

Corollary 1. *Positive system Σ_0 with $q = 1$ is asymptotically stable and $\|G\|_\infty < \gamma$ if and only if there exists $P \in \mathbb{D}_+^{n\times n}$ such that*

$$\begin{bmatrix} (A+A_1)^TP(A+A_1)-P+(C+C_1)^T(C+C_1) & (A+A_1)^TPB+(C+C_1)^TD \\ B^TP(A+A_1)+D^T(C+C_1) & B^TPB+D^TD-\gamma^2I \end{bmatrix} \prec 0. \tag{13}$$

Remark 1. *From Lemma 4 and Theorem 1, it is clear that the exact value of $\|G_0\|_\infty$ is given by $\bar{\sigma}(G_0(1))$ if positive system Σ_0 with $q = 1$ is asymptotically stable. That is, H_∞ norm of system Σ_0 is equivalent to that of system Σ_1, which is independent of time delays. Due to this fact, Theorem 1 can be easily extended to the case of multiple time delays, that is, $q > 1$, and A_1 and C_1 being replaced by $\sum_{i=1}^q A_i$ and $\sum_{i=1}^q C_i$.*

The BRL for positive system Σ_0 with multiple time delays can be directly obtained in the following theorem, in which $\tilde{A}$ and $\tilde{C}$ are given in system Σ_1.

Theorem 2. *[**Multiple delay**] Positive system Σ_0 with $q > 1$ is asymptotically stable and $\|G_0\|_\infty < \gamma$ if and only if there exists a matrix $P \in \mathbb{D}_+^{n\times n}$ such that*

$$\begin{bmatrix} \tilde{A}^TP\tilde{A}-P+\tilde{C}^T\tilde{C} & \tilde{A}^TPB+\tilde{C}^TD \\ B^TP\tilde{A}+D^T\tilde{C} & B^TPB+D^TD-\gamma^2I \end{bmatrix} \prec 0. \tag{14}$$

4 DYNAMIC OUTPUT FEEDBACK H_∞ CONTROL

Due to the fact that full access to the system state is usually impossible in real plants and often only partial information of the state can be measured, one has to use a controller based on output measurements. It becomes necessary to develop H_∞ control theory via output feedback control signal. On the basis of the above preparatory work, next an explicit delay-independent characterization of the positivity preserving H_∞ control will be developed. Since H_∞ norms of positive time-delay systems only depend on system matrices, our attention is restricted to the case of single delay (that is, $q = 1$) and then the derived results can be easily extended to the case of multiple delays (that is, $q > 1$).

Consider a discrete-time positive system with one constant delay as follows

$$\begin{aligned} x(k+1) &= Ax(k)+A_1x(k-d)+B\omega(k)+B_1u(k),\\ z(k) &= Cx(k)+C_1x(k-d)+D\omega(k)+B_2u(k),\\ y(k) &= Fx(k)+H\omega(k),\\ x(k) &= \phi(k),\ k\in[-d,0], \end{aligned} \tag{15}$$

where $x(k) \in \mathbb{R}^n$ is the state, $\omega(k) \in \mathbb{R}^m$ is the exogenous input, $u(k) \in \mathbb{R}^l$ is the control input, $z(k) \in \mathbb{R}^p$ is the controlled output, $y(k) \in \mathbb{R}^r$ is the measurement. $A, A_1, B, B_1, B_2, C, C_1, D, F, H$ are real matrices with compatible dimensions. A dynamic output feedback controller is given by

$$\begin{aligned} \xi(k+1) &= A_K \xi(k) + B_K y(k), \\ u(k) &= C_K \xi(k) + D_K y(k), \end{aligned} \tag{16}$$

where $\xi(k) \in \mathbb{R}^r$ is the controller state, A_K, B_K, C_K, D_K are the controller gain matrices to be designed. The following closed-loop system is conducted from system (15) via the output feedback controller (16) .

$$\begin{aligned} \begin{bmatrix} x(k+1) \\ \xi(k+1) \end{bmatrix} &= \begin{bmatrix} A+B_1D_KF & B_1C_K \\ B_KF & A_K \end{bmatrix} \begin{bmatrix} x(k) \\ \xi(k) \end{bmatrix} \\ &+ \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(k-d) \\ \xi(k-d) \end{bmatrix} + \begin{bmatrix} B+B_1D_KH \\ B_KH \end{bmatrix} \omega(k), \\ z(k) &= \begin{bmatrix} C+B_2D_KF & B_2C_K \end{bmatrix} \begin{bmatrix} x(k) \\ \xi(k) \end{bmatrix} \\ &+ \begin{bmatrix} C_1 & 0 \end{bmatrix} \begin{bmatrix} x(k-d) \\ \xi(k-d) \end{bmatrix} + (D+B_2D_KH)\omega(k), \end{aligned} \tag{17}$$

The validity of the performance-based design lies in whether the closed-loop performance requirement can be satisfied easily. Let us take an exploration of H_∞ performance-based control design upon the BRL representation through an output feedback control, which allows that the closed-loop system is positive, asymptotically stable and $\|G\|_\infty < \gamma$.

Theorem 3. *Given positive system* (15) *and a constant scalar $\gamma > 0$, the existence of a dynamic output feedback controller* (16) *such that the closed-loop system* (17) *is positive, asymptotically stable and $\|G\|_\infty < \gamma$, is equivalent to the existence of matrices $P_1 \in \mathbb{D}_+^{n\times n}, P_2 \in \mathbb{D}_+^{s\times s}, Q_1 \in \mathbb{D}_+^{n\times n}, Q_2 \in \mathbb{D}_+^{s\times s}$, $L_1 \in \mathbb{R}^{s\times s}$, $L_2 \in \mathbb{R}^{s\times r}$, $L_3 \in \mathbb{R}^{l\times s}$, $L_4 \in \mathbb{R}^{l\times r}$ satisfying $P_1Q_1 = I$, $P_2Q_2 = I$, $A+B_1L_4F \geq 0$, $B_1L_3 \geq 0$, $L_2F \geq 0$, $L_1 \geq 0$, $B+B_1L_4H \geq 0$, $L_2H \geq 0$, $C+B_2L_4F \geq 0$, $B_2L_3 \geq 0$, $D+B_2L_4H \geq 0$, and*

$$\begin{bmatrix} -P_1 & 0 & 0 & A+A_1+B_1L_4F & B_1L_3 & B+B_1L_4H \\ * & -P_2 & 0 & L_2F & L_1 & L_2H \\ * & * & -I & C+C_1+B_2L_4F & B_2L_3 & D+B_2L_4H \\ * & * & * & -Q_1 & 0 & 0 \\ * & * & * & * & -Q_2 & 0 \\ * & * & * & * & * & -\gamma^2 I \end{bmatrix} \prec 0. \tag{18}$$

Then the desired controller gain matrices are given by

$$A_K = L_1, B_K = L_2, C_K = L_3, D_K = L_4.$$

Proof: Applying Schur complement lemma and setting $P_1^{-1} = Q_1$, $P_2^{-1} = Q_2$, $A_K = L_1$, $B_K = L_2$, $C_K = L_3$, $D_K = L_4$, inequality (18) and two equality constraints can be derived from Theorem 2. Other inequalities used for guaranteeing the positivity of the closed-loop system (17) can be derived directly from Lemma 2. The proof is completed. □

5 ROBUST H_∞ CONTROL

It is noted that, for two Schur stable matrices $A_1 \geq 0$ and $A_2 \geq 0$ with $A_1 \geq A_2$, we have $A_1^{-1} \geq A_2^{-1}$. Motivated by this fact, there is a possible extension of Theorem 1 to uncertain time-delay positive systems. In this section, consider an interval uncertain discrete-time positive system with a time delay as follows

$$\begin{aligned} & x(k+1) = A^I x(k) + A_1^I x(k-d) + B^I \omega(k) + B_1 u(k), \\ & z(k) = C^I x(k) + C_1^I x(k-d) + D^I \omega(k) + B_2 u(k), \\ & y(k) = Fx(k) + H\omega(k), \\ & x(k) = \phi(k), \ k \in [-d, 0], \end{aligned} \tag{19}$$

where $A^I \in [\underline{A}, \overline{A}]$, $A_1^I \in [\underline{A}_1, \overline{A}_1]$, $B^I \in [\underline{B}, \overline{B}]$, $C^I \in [\underline{C}, \overline{C}]$, $C_1^I \in [\underline{C}_1, \overline{C}_1]$, $D^I \in [\underline{D}, \overline{D}]$, $\underline{A} \geq 0$, $\underline{A}_1 \geq 0$, $\underline{B} \geq 0$, $\underline{C} \geq 0$, $\underline{C}_1 \geq 0$, $\underline{D} \geq 0$,and $\underline{A}$, $\overline{A}$, $\underline{A}_1$, $\overline{A}_1$, $\underline{B}$, $\overline{B}$, $\underline{C}$, $\overline{C}$, $\underline{C}_1$, $\overline{C}_1$, $\underline{D}$, $\overline{D}$ are all constrained in metric space.

Theorem 4. *Interval uncertain positive system* (19) *is robustly asymptotically stable and $\|G\|_\infty < \gamma$ if and only if there exists a matrix $P \in \mathbb{D}_+^{n\times n}$ satisfying that*

$$\begin{bmatrix} (\overline{A}+\overline{A}_1)^T P(\overline{A}+\overline{A}_1) - P + (\overline{C}+\overline{C}_1)^T(\overline{C}+\overline{C}_1) & (\overline{A}+\overline{A}_1)^T P\overline{B} + (\overline{C}+\overline{C}_1)^T \overline{D} \\ \overline{B}^T P(\overline{A}+\overline{A}_1) + \overline{D}^T(\overline{C}+\overline{C}_1) & \overline{B}^T P\overline{B} + \overline{D}^T\overline{D} - \gamma^2 I \end{bmatrix} \prec 0.$$

Proof: If system (19) is positive and robustly asymptotically stable, $\rho(A^I) < 1$ for any $A^I \in [\underline{A}, \overline{A}]$. It follows that $C^I + C_1^I(I - A^I - A_1^I)^{-1}B^I + D^I \leq \overline{C} + \overline{C}_1(I - \overline{A} - \overline{A}_1)^{-1}\overline{B} + \overline{D}$. Furthermore, from Lemma 2, one gets

$$\begin{aligned} & \left\| C^I + C_1^I(I - A^I - A_1^I)^{-1}B^I + D^I \right\|_\infty \\ \leq & \left\| \overline{C} + \overline{C}_1(I - \overline{A} - \overline{A}_1)^{-1}\overline{B} + \overline{D} \right\|_\infty < \gamma. \end{aligned}$$

Therefore, according to Theorem 1, sufficiency and necessity conditions are obvious. □

Next, our objective is to design a dynamic output feedback controller in (16) such that the following closed-loop system

$$
\begin{aligned}
\begin{bmatrix} x(k+1) \\ \xi(k+1) \end{bmatrix} &= \begin{bmatrix} A^I+B_1D_KF & B_1C_K \\ B_KF & A_K \end{bmatrix}\begin{bmatrix} x(k) \\ \xi(k) \end{bmatrix} \\
&+\begin{bmatrix} A_1^I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x(k-d) \\ \xi(k-d) \end{bmatrix}+\begin{bmatrix} B^I+B_1D_KH \\ B_KH \end{bmatrix}\omega(k), \\
z(t) &= \begin{bmatrix} C^I+B_2D_KF & B_2C_K \end{bmatrix}\begin{bmatrix} x(k) \\ \xi(k) \end{bmatrix} \\
&+\begin{bmatrix} C_1^I & 0 \end{bmatrix}\begin{bmatrix} x(k-d) \\ \xi(k-d) \end{bmatrix}+(D^I+B_2D_KH)\,\omega(k).
\end{aligned} \tag{20}
$$

is positive, robustly asymptotically stable and $||G||_\infty < \gamma$. □

Theorem 5. *Given positive system* (19) *and a scalar* $\gamma > 0$, *there exists a dynamic output feedback controller* (16) *such that the closed-loop system* (20) *is positive, robustly asymptotically stable and* $||G_0||_\infty < \gamma$ *if and only if there exist matrices* $P_1 \in \mathbb{D}_+^{n\times n}, P_2 \in \mathbb{D}_+^{s\times s}, Q_1 \in \mathbb{D}_+^{n\times n}, Q_2 \in \mathbb{D}_+^{s\times s}$, $L_1 \in \mathbb{R}^{s\times s}$, $L_2 \in \mathbb{R}^{s\times r}$, $L_3 \in \mathbb{R}^{l\times s}$, $L_4 \in \mathbb{R}^{l\times r}$ *such that* $\underline{A}+B_1L_4F \geq 0$, $B_1L_3 \geq 0$, $L_2F \geq 0$, $L_1 \geq 0$, $\underline{B}+B_1L_4H \geq 0$, $L_2H \geq 0$, $\underline{C}+B_2L_4F \geq 0$, $B_2L_3 \geq 0$, $\underline{D}+B_2L_4H \geq 0$, $P_1Q_1 = I$, $P_2Q_2 = I$, *and*

$$
\begin{bmatrix}
-P_1 & 0 & 0 & \overline{A}+\overline{A}_1+B_1L_4F & B_1L_3 & \overline{B}+B_1L_4H \\
* & -P_2 & 0 & L_2F & L_1 & L_2H \\
* & * & -I & \overline{C}+\overline{C}_1+B_2L_4F & B_2L_3 & \overline{D}+B_2L_4H \\
* & * & * & -Q_1 & 0 & 0 \\
* & * & * & * & -Q_2 & 0 \\
* & * & * & * & * & -\gamma^2 I
\end{bmatrix} \prec 0. \tag{21}
$$

If the above conditions hold, then the desired controller gain matrices are given by

$$A_K = L_1, B_K = L_2, C_K = L_3, D_K = L_4.$$

Proof: It is obvious that $\underline{A}+B_1D_KF \leq A^I+B_1D_KF$ for any $A^I \in [\underline{A},\overline{A}]$. If $\underline{A}+B_1L_4F \geq 0$ holds, then $A^I+B_1L_4F \geq 0$. Setting $P_1^{-1} = Q_1$, $P_2^{-1} = Q_2$, $A_K = L_1$, $B_K = L_2$, $C_K = L_3$, $D_K = L_4$, and taking a similar line as the proof of Theorem 3, the detailed proof is omitted. □

6 NUMERICAL EXAMPLE

This section presents one numerical example to illustrate the effectiveness of the proposed results. Consider a discrete-time interval uncertain positive system in (19) with one delay in the system state, and system matrices given as follows:

$$
\underline{A} = \begin{bmatrix} 0.3648 & 0.3986 & 0.2695 \\ 0.3178 & 0.4146 & 0.4423 \\ 0.4812 & 0.1218 & 0.3757 \end{bmatrix}, \overline{A} = \begin{bmatrix} 0.3931 & 0.4181 & 0.2914 \\ 0.3398 & 0.4527 & 0.4718 \\ 0.5054 & 0.1470 & 0.3990 \end{bmatrix},
$$

$$
\underline{A}_1 = \begin{bmatrix} 0.0221 & 0.0982 & 0.0322 \\ 0.0313 & 0.1001 & 0.0271 \\ 0.0182 & 0.0235 & 0.0283 \end{bmatrix}, \overline{A}_1 = \begin{bmatrix} 0.0323 & 0.1002 & 0.0385 \\ 0.0348 & 0.1320 & 0.0334 \\ 0.0293 & 0.0264 & 0.0379 \end{bmatrix},
$$

$$
\underline{B} = \begin{bmatrix} 0.1560 \\ 0.1820 \\ 0.141 \end{bmatrix}, \overline{B} = \begin{bmatrix} 0.1835 \\ 0.2273 \\ 0.1705 \end{bmatrix}, B_1 = \begin{bmatrix} 0.2318 \\ 0.4836 \\ 0.1931 \end{bmatrix}, B_2 = 0.2358,
$$

$$
\underline{C} = \begin{bmatrix} 0.2170 & 0.1911 & 0.2143 \end{bmatrix}, \overline{C} = \begin{bmatrix} 0.2321 & 0.2233 & 0.3097 \end{bmatrix},
$$

$$
\underline{C}_1 = \begin{bmatrix} 0.0243 & 0.0435 & 0.0219 \end{bmatrix}, \overline{C}_1 = \begin{bmatrix} 0.0339 & 0.0500 & 0.0254 \end{bmatrix},
$$

$$
F = \begin{bmatrix} 0.1408 & 0.1619 & 0.2045 \end{bmatrix}, \underline{D} = 0.2970, \overline{D} = 0.3102.
$$

It can be verified that this system is not robustly stable. We now apply the proposed approach to find a reduced-order dynamic output feedback controller in (16) with $r = 2$ such that the closed-loop system is positive, robustly asymptotically stable and $||G||_\infty < 1$. One group of feasible solutions of the constrained conditions in Theorem 3 is obtained as follows,

$$
P_1 = \begin{bmatrix} 0.9972 & 0 & 0 \\ 0 & 1.2481 & 0 \\ 0 & 0 & 0.8515 \end{bmatrix}, Q_1 = \begin{bmatrix} 1.0028 & 0 & 0 \\ 0 & 0.8012 & 0 \\ 0 & 0 & 1.1744 \end{bmatrix},
$$

$$
P_2 = \begin{bmatrix} 0.9733 & 0 \\ 0 & 0.9733 \end{bmatrix}, Q_2 = \begin{bmatrix} 1.0275 & 0 \\ 0 & 1.0275 \end{bmatrix},
$$

$$
L_1 = \begin{bmatrix} 0.1724 & 0.1724 \\ 0.1724 & 0.1724 \end{bmatrix}, L_2 = \begin{bmatrix} 0.0423 \\ 0.0423 \end{bmatrix},
$$

$$
L_3 = \begin{bmatrix} 0.0363 & 0.0363 \end{bmatrix}, L_4 = -3.8956.
$$

The desired controller gain matrices A_K, B_K, C_K and D_K are given by L_1, L_2, L_3, L_4, respectively. Figure 1 gives the maximal singular value plots of the closed-loop system when $d = 2$, $d = 50$, and $d = 0$. It shows clearly in this example that H_∞ norm of discrete-time interval uncertain positive system with time delays is independent of the delay magnitude.

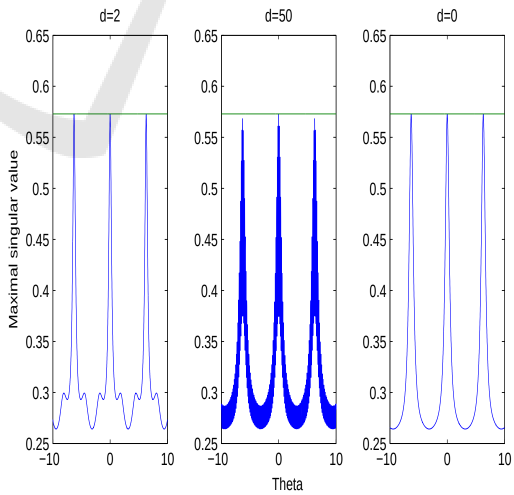

Figure 1: Maximal singular value plots of the closed-loop system.

7 CONCLUSIONS

This paper has established the BRL for discrete-time positive linear system with multiple time delays. The proposed delay-independent criteria results reveal that H_∞ performance of positive systems with time delays in state and output equations is equivalent to the characterization of the corresponding delay-free systems. The necessary and sufficient conditions in the forms of matrix (in)equalities are established for the H_∞ control problem via dynamic output feedback controls, which can be easily solved by using Matlab toolbox, although the proposed approach is not guaranteed to find a feasible solution even it exists.

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