Lyapunov Functions for Linear Stochastic Differential Equations: BMI Formulation of the Conditions

Sigurdur Hafstein
Science Institute and Faculty of Physical Sciences, University of Iceland, Dunhagi 5, 107 Reykjavík, Iceland

Keywords: Stochastic Differential Equation, Lyapunov Function, Bilinear Matrix Inequalities.

Abstract: We present a bilinear matrix inequality (BMI) formulation of the conditions for a Lyapunov functions for autonomous, linear stochastic differential equations (SDEs). We review and collect useful results from the theory of stochastic stability of the null solution of an SDE. Further, we discuss the Itô- and Stratonovich interpretation and linearizations and Lyapunov functions for linear SDEs. Then we discuss the construction of Lyapunov functions for the damped pendulum, wihere the spring constant is modelled as a stochastic process. We implement in Matlab the characterization of its canonical Lyapunov function as BMI constraints and consider some practical implementation strategies. Further, we demonstrate that the general strategy is applicable to general autonomous and linear SDEs. Finally, we verify our findings by comparing with results from the literature.

1 INTRODUCTION

For a linear deterministic system $\dot{\mathbf{x}} = A\mathbf{x}$, $A \in \mathbb{R}^{d\times d}$, with a globally asymptotically stable (GAS) equilibrium, one can compute a quadratic Lyapunov function $V(\mathbf{x}) = \mathbf{x}^T Q\mathbf{x}$ by solving a particular type of the Sylvester equation (Sylvester, 1884), the so-called continuous-time Lyapunov equation,

$$Q^T A + AQ = -P, \qquad (1)$$

for any given symmetric and (strictly) positive definite P. In the spirit of linear matrix inequalities (LMI) (Boyd et al., 1994) this is sometimes written:

$$\text{compute } Q \succ 0, \text{ such that } Q^T A + AQ \prec 0,$$

although there is no need to solve an LMI problem because (1) is just a linear equation that has a symmetric and positive definite solution Q for any given symmetric and positive definite P, if and only if A is Hurwitz (the real-parts of all eigenvalues are strictly less than zero). For efficient algorithms to solve it cf. e.g. (Bartels and Stewart, 1972; Mikkelsen, 2009). However, if one wants to compute a common quadratic Lyapunov function $V(\mathbf{x}) = \mathbf{x}^T Q\mathbf{x}$ for a collection of linear systems $\dot{\mathbf{x}} = A_i\mathbf{x}$, $i = 1,2,\ldots,m$, cf. e.g. (Filippov, 1988; Aubin and Cellina, 1984; Liberzon, 2003; Sun and Ge, 2011), one must resort to solve the LMI:

$$Q \succ 0 \text{ and } Q^T A_i + A_i Q \prec 0 \text{ for } i = 1,2,\ldots,m.$$

The stability theory of stochastic differential equations (SDEs) is a much less mature subject and algorithms or general procedures to compute Lyapunov functions, even for autonomous, linear SDEs, are not available. In this paper we will show that there is a canonical form for such functions and we will formulate the conditions for a Lyapunov function as a bilinear matrix inequality (BMI) feasibility problem (VanAntwerp and Braatz, 2000).

The organization of the paper is as follows: after fixing the notation we recall the basic theory of SDEs, their stability, and Lyapunov functions for SDEs in Section 2. In Section 3 we study the autonomous, linear case in more detail, before we derive our BMI problem in Section 4, both in general and, as a demonstration, for an important example from the literature. In Section 5 we give a short verification of our results and then conclude the paper with some finishing remarks on future research. We give several code examples of how to efficiently define the BMI problem, but we do not consider in this paper how to solve it. Computing a solution to a BMI problem is a challenging task, cf. e.g. (Kheirandishfard et al., 2018b; Kheirandishfard et al., 2018a) that will be tackled for our particular BMI problem in the future.

Notation: We denote by $\|\mathbf{x}\|$ the Euclidian norm of a vector $\mathbf{x} \in \mathbb{R}^d$. Vectors are assumed to be column vectors. For a symmetric matrix $Q \in \mathbb{R}^{d\times d}$ we write $Q \succ 0$ and $Q \succeq$ if Q is (strictly) positive definite and

Hafstein, S.
Lyapunov Functions for Linear Stochastic Differential Equations: BMI Formulation of the Conditions.
DOI: 10.5220/0008192201470155
In *Proceedings of the 16th International Conference on Informatics in Control, Automation and Robotics (ICINCO 2019)*, pages 147-155
ISBN: 978-989-758-380-3

semi-positive definite, respectively, and similarly for $\prec$ and $\preceq$. For a symmetric $Q \succ 0$ we define the energetic norm $\|\mathbf{x}\|_Q := \sqrt{\mathbf{x}^T Q \mathbf{x}}$. We write $\mathbb{P}$ and $\mathbb{E}$ for probability and expectation respectively. The underlying probability spaces should always be clear from the context. The abbreviation *a.s.* stands for *almost surely*, i.e. with probability one, and $\stackrel{\text{a.s.}}{=}$ means equal a.s.

2 SDEs AND LYAPUNOV STABILITY

In this section we describe the class of problems we are concerned with and recall and collect some definitions and useful results. For a more detailed description of the problems cf. (Gudmundsson and Hafstein, 2018, §2) and the books (Mao, 2008) and (Khasminskii, 2012), to which we will frequently refer.

Stochastic differential equations (SDEs) are ordinary differential equations with some added random distribution or noise. The noise is usually assumed to be "white noise" and is modelled with a U-dimensional Wiener process $\mathbf{W} = (W_1, W_2, \ldots, W_U)^T$, where the W_i, $i = 1, 2, \ldots, U$, are independent 1-dimensional Wiener processes. The general autonomous d-dimensional SDE of Itô type is given by

$$\mathrm{d}\mathbf{X}(t) = \mathbf{f}(\mathbf{X}(t))\mathrm{d}t + \mathbf{g}(\mathbf{X}(t)) \cdot \mathrm{d}\mathbf{W}(t)$$

or equivalently

$$\mathrm{d}\mathbf{X}(t) = \mathbf{f}(\mathbf{X}(t))\mathrm{d}t + \sum_{u=1}^{U} \mathbf{g}^u(\mathbf{X}(t)) \cdot \mathrm{d}W_u(t) \qquad (2)$$

for $i = 1, 2, \ldots, d$. Thus $\mathbf{f} = (f_1, f_2, \ldots, f_d)^T$, $\mathbf{g} = (\mathbf{g}^1, \mathbf{g}^2, \ldots, \mathbf{g}^U)$, and $\mathbf{g}^u = (g_1^u, g_2^u, \ldots, g_d^u)^T$, where $f_i, g_i^u : \mathbb{R}^d \to \mathbb{R}$. We consider *strong solutions* to the SDE (2), which are given by

$$\mathbf{X}^{\mathbf{x}}(t) = \mathbf{x} + \int_0^t \mathbf{f}(\mathbf{X}(s))\mathrm{d}s + \int_0^t \mathbf{g}(\mathbf{X}(s))\mathrm{d}\mathbf{W}(s)$$

for deterministic initial value solutions $\mathbf{X}^{\mathbf{x}}(t)$ fulfilling $\mathbf{X}^{\mathbf{x}}(0) = \mathbf{x} \in \mathbb{R}^d$ a.s., where the second integral is interpreted in the Itô sense.

Another much used type of SDEs is given by the Stratonovich approach

$$\mathrm{d}\mathbf{X}(t) = \mathbf{f}(\mathbf{X}(t))\mathrm{d}t + \mathbf{g}(\mathbf{X}(t)) \circ \mathrm{d}\mathbf{W}(t), \qquad (3)$$

with strong solutions for deterministic initial values $\mathbf{X}^{\mathbf{x}}(0) = \mathbf{x} \in \mathbb{R}^d$ a.s. given by

$$\mathbf{X}^{\mathbf{x}}(t) = \mathbf{x} + \int_0^t \mathbf{f}(\mathbf{X}(s))\mathrm{d}s + \int_0^t \mathbf{g}(\mathbf{X}(s)) \circ \mathrm{d}\mathbf{W}(s),$$

where the second integral is interpreted in the Stratonovich sense.

It is adequate or even instrumental to use Itô's stochastic integral if the noise has no autocorrelation. This is the case, for example, when modelling financial markets. When the noise is a model of some unknown or complicated dynamic sub-system, then it is more naturally represented using the Stratonovich approach. Fortunately, for our purposes it is enough to study the SDE (2), because the Stratonovich SDE (3) is equivalent to the Itô type SDE

$$\mathrm{d}\mathbf{X}(t) = \widetilde{\mathbf{f}}(\mathbf{X}(t))\mathrm{d}t + \sum_{u=1}^{U} \mathbf{g}^u(\mathbf{X}(t)) \cdot \mathrm{d}W_u(t), \qquad (4)$$

with

$$\widetilde{\mathbf{f}}(\mathbf{x}) := \mathbf{f}(\mathbf{x}) + \frac{1}{2} \sum_{u=1}^{U} D\mathbf{g}^u(\mathbf{x})\mathbf{g}^u(\mathbf{x}),$$

where $D\mathbf{g}^u$ is the Jacobian of $\mathbf{g}^u$. The term added to $\mathbf{f}(\mathbf{x})$ is often referred to as "noise-induced drift" for obvious reasons.

Hence, we will in the following concentrate on the Itô SDE (2), but we will also interpret some of our results for the Stratonovich SDE (3) too. Further, we will from now on assume that the origin is an equilibrium of the system (2), i.e. $\mathbf{f}(\mathbf{0}) = \mathbf{0}$ and $\mathbf{g}^{\mathbf{u}}(\mathbf{0}) = \mathbf{0}$ for $u = 1, 2, \ldots, U$. Note that if the origin is an equilibrium of the Stratonovich SDE (3), then it is also an equilibrium of the equivalent Itô SDE (4). As shown in (Mao, 2008) it suffices to consider deterministic initial value solutions when studying the stability of an equilibrium.

A great number of concepts are used for classifying stability of equilibria of SDEs and the nomenclature is far from uniform. For our needs *global asymptotic stability in probability* of the zero solution (Khasminskii, 2012, (5.15)), also referred to as *stochastic asymptotic stability in the large* (Mao, 2008, Definition 4.2.1 (iii)), is the most useful. However, we need many other concepts in intermediate steps. For a more detailed discussion of the stability of SDEs see the books by Khasminskii (Khasminskii, 2012) or Mao (Mao, 2008).

We recall a few definitions of stability that we use later:

Definition 2.1 (Stability in Probability (*SiP*)). *The null solution* $\mathbf{X}(t) \stackrel{a.s.}{=} \mathbf{0}$ *to the SDE* (2) *is said to be* stable in probability *(SiP) if for every* $r > 0$ *and* $0 < \varepsilon < 1$ *there exists a* $\delta > 0$ *such that*:

$$\|\mathbf{x}\| \leq \delta \textit{ implies } \mathbb{P}\left\{\sup_{t \geq 0} \|\mathbf{X}^{\mathbf{x}}(t)\| \leq r\right\} \geq 1 - \varepsilon.$$

□

Definition 2.2 (Asymptotic Stability in Probability (*ASiP*)). *The null solution* $\mathbf{X}(t) \stackrel{a.s.}{=} \mathbf{0}$ *to the SDE* (2) *is said to be* asymptotically stable in probability *(ASiP)* *if it is SiP and in addition for every* $0 < \varepsilon < 1$ *there exists a* $\delta > 0$ *such that*:

$$\|\mathbf{x}\| \leq \delta \text{ implies } \mathbb{P}\left\{\lim_{t\to\infty} \|\mathbf{X}^{\mathbf{x}}(t)\| = 0\right\} \geq 1 - \varepsilon.$$

□

Definition 2.3 (Global Asymptotic Stability in Probability (*GASiP*)). *The null solution* $\mathbf{X}(t) \overset{a.s.}{=} \mathbf{0}$ *to the SDE* (2) *is said to be* globally asymptotically stable in probability *(GASiP) if it is SiP and for any* $\mathbf{x} \in \mathbb{R}^d$ *we have*

$$\mathbb{P}\left\{\lim_{t\to\infty} \|\mathbf{X}^{\mathbf{x}}(t)\| = 0\right\} = 1.$$

□

Definition 2.4 (Exponential p-Stability (p-*ES*)). *The null solution* $\mathbf{X}(t) \overset{a.s.}{=} \mathbf{0}$ *to the SDE* (2) *is said to be* exponentially p-stable *(p-ES) for a constant* $p > 0$, *if there exist constants* $A, \alpha > 0$ *such that*

$$\mathbb{E}\{\|\mathbf{X}^{\mathbf{x}}(t)\|^p\} \leq A\|\mathbf{x}\|^p e^{-\alpha t}$$

holds true for all $\mathbf{x} \in \mathbb{R}^d$ *and all* $t \geq 0$.

Since we are concerned with autonomous SDEs the concepts of SiP and GASiP automatically include *uniform stability in probability* and *stability in the large uniformely in* $t > 0$ respectively, cf. (Khasminskii, 2012, Remark 5.3, §6.5). The conditions for SiP and ASiP are more commonly stated

$$\lim_{\|\mathbf{x}\|\to 0} \mathbb{P}\{\sup_{t>0} \|\mathbf{X}^{\mathbf{x}}(t)\| \leq r\} = 1 \quad \text{for all } r > 0$$

for SiP and additionally

$$\lim_{\|\mathbf{x}\|\to 0} \mathbb{P}\left\{\limsup_{t\to\infty} \|\mathbf{X}^{\mathbf{x}}(t)\| = 0\right\} = 1$$

for ASiP. This is clearly equivalent to our definitions as can be seen by fixing $r > 0$ and writing down the definition of a limit: *for every* $\varepsilon > 0$ *there exists a* $\delta > 0$. Our formulation is more natural when studying the stochastic counterpart of the *basin of attraction* (BOA) in the stability theory for deterministic systems, cf. (Gudmundsson and Hafstein, 2018). Instead of the limit $\|\mathbf{x}\| \to 0$ we consider: Given some *confidence* $0 < \gamma \leq 1$ how far from the origin can sample paths start and still approach the equilibrium as $t \to \infty$ with probability greater than or equal to γ. The exact definition is:

Definition 2.5 (γ-Basin Of Attraction (γ-BOA)). *Consider the system* (2) *and let* $0 < \gamma \leq 1$. *The set*

$$\left\{\mathbf{x} \in \mathbb{R}^d : \mathbb{P}\left\{\lim_{t\to\infty} \|\mathbf{X}^{\mathbf{x}}(t)\| = 0\right\} \geq \gamma\right\} \quad (\gamma\text{-BOA})$$

is called the γ-basin of attraction *(γ-BOA) of the equilibrium at the origin.* □

Any sample path started in γ-BOA will tend towards the origin with probability γ or greater.

Just as for deterministic differential equations, the various stability properties for SDEs can be characterized by the existence of so-called Lyapunov functions. The stochastic counterpart to the orbital derivative of $V : \mathbb{R}^d \to \mathbb{R}$ for deterministic systems, i.e. the derivative along solution trajectories of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x}(0) = \xi$, given by

$$\left.\frac{d}{dt} V(\phi(t,\xi))\right|_{t=0} = \nabla V(\xi) \cdot \mathbf{f}(\xi),$$

is the *generator*, where one has to add a term accounting for the stochastic drift:

Definition 2.6 (Generator of an SDE). *For the SDE* (2) *the associated* generator *for some appropriately differentiable* $V : \mathcal{U} \to \mathbb{R}$ *with* $\mathcal{U} \subset \mathbb{R}^d$ *is given by*

$$LV(\mathbf{x}) := \nabla V(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) + \frac{1}{2} \sum_{i,j=1}^{d} \sum_{u=1}^{U} g_i^u(\mathbf{x}) g_j^u(\mathbf{x}) \frac{\partial^2 V}{\partial x_i \partial x_j}(\mathbf{x}).$$

□

Note that LV is the drift term in the expression for the stochastic differential of the process $t \mapsto V(\mathbf{X}^{\mathbf{x}}(t))$. Accordingly, the Stratonovich SDE (3) has the generator

$$LV(\mathbf{x}) := \nabla V(\mathbf{x}) \cdot \left[\mathbf{f}(\mathbf{x}) + \frac{1}{2} \sum_{u=1}^{U} D\mathbf{g}^u(\mathbf{x}) \mathbf{g}^u(\mathbf{x})\right] + \frac{1}{2} \sum_{i,j=1}^{d} \sum_{u=1}^{U} g_i^u(\mathbf{x}) g_j^u(\mathbf{x}) \frac{\partial^2 V}{\partial x_i \partial x_j}(\mathbf{x}). \quad (5)$$

When defining Lyapunov functions for SDEs it is convenient to introduce the function class $\mathcal{K}_\infty$ of strictly increasing continuous functions $\mu : \mathbb{R} \to \mathbb{R}$, such that $\mu(0) = 0$ and $\lim_{x\to\infty} \mu(x) = \infty$.

Definition 2.7 (Lyapunov function for a SDE). *Consider the system* (2). *A function*

$$V \in C(\mathcal{U}) \cap C^2(\mathcal{U} \setminus \{\mathbf{0}\}),$$

where $\mathbf{0} \in \mathcal{U} \subset \mathbb{R}^d$ *is a domain, is called a (local) Lyapunov function for the the system* (2) *if there are functions* $\mu_1, \mu_2, \mu_3 \in \mathcal{K}_\infty$, *such that V fulfills the properties*:

(i) $\mu_1(\|\mathbf{x}\|) \leq V(\mathbf{x}) \leq \mu_2(\|\mathbf{x}\|)$ *for all* $\mathbf{x} \in \mathcal{U}$

(ii) $LV(\mathbf{x}) \leq -\mu_3(\|\mathbf{x}\|)$ *for all* $\mathbf{x} \in \mathcal{U} \setminus \{\mathbf{0}\}$

If $\mathcal{U} = \mathbb{R}^d$ *the function V is said to be a global Lyapunov function.*

It is instrumental that one does not demand that a Lyapunov function V for an SDE is differentiable at the origin, cf. (Khasminskii, 2012, Remark 5.5);

see also (Bjornsson and Hafstein, 2018) for similar results for *almost sure exponential stability* introduced in (Mao, 2008, Def. 4.3.3.1).

We conclude this section with the following general theorem:

Theorem 2.8. *If there exists a Lyapunov functions for the system* (2)*, then the null solution is ASiP. Further, let* $V_{\max} > 0$ *and assume that* $V^{-1}([0,V_{\max}])$ *is a compact subset of the domain* $\mathcal{U}$ *of the Lyapunov function. Then, for every* $0 < \beta < 1$ *the set* $V^{-1}([0,\beta V_{\max}])$ *is a subset of the* $(1-\beta)$*-BOA of the origin. In particular, if the Lyapunov function is global, i.e.* $\mathcal{U} = \mathbb{R}^d$*, then the origin is GASiP.*

For these results see (Bjornsson et al., 2018, Th. 2.6) and (Khasminskii, 2012, Th. 5.5, Cor. 5.1).

3 LINEAR AUTONOMOUS SDEs

We will now consider equations (2) and (3) assuming that $\mathbf{f}$ and the $\mathbf{g}^u$ are linear, i.e. there exists matrices $F, G^u \in \mathbb{R}^{d\times d}$ such that $\mathbf{f}(\mathbf{x}) = F\mathbf{x}$ and $\mathbf{g}^u(\mathbf{x}) = G^u\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^d$ and $u = 1,2,\dots,U$. The SDE (2) can then be written

$$d\mathbf{X}(t) = F\mathbf{X}(t)dt + \sum_{u=1}^{U} G^u\mathbf{X}(t)\cdot dW_u(t) \qquad (6)$$

and its generator, with F_{ij} and G^u_{ij} as the components of the matrices F and G^u respectively, as

$$\begin{aligned} LV(\mathbf{x}) &= \sum_{i,j=1}^{d} F_{ij}x_j \frac{\partial V}{\partial x_i}(\mathbf{x}) \qquad (7)\\ &+ \frac{1}{2}\sum_{i,j,k,\ell=1}^{d}\sum_{u=1}^{U} x_k x_\ell G^u_{ik}G^u_{j\ell}\frac{\partial^2 V}{\partial x_i \partial x_j}(\mathbf{x}). \end{aligned}$$

Similarly, for the Stratonovich SDE (3) the generator is given by

$$\begin{aligned} LV(\mathbf{x}) &= \sum_{i,j=1}^{d}\left(F_{ij}x_j + \frac{1}{2}\sum_{k=1}^{d}\sum_{u=1}^{U} G^u_{ij}G^u_{jk}x_k\right)\frac{\partial V}{\partial x_i}(\mathbf{x})\\ &+ \frac{1}{2}\sum_{i,j,k,\ell=1}^{d}\sum_{u=1}^{U} x_k x_\ell G^u_{ik}G^u_{j\ell}\frac{\partial^2 V}{\partial x_i \partial x_j}(\mathbf{x}). \qquad (8) \end{aligned}$$

For the autonomous linear SDE with constant coefficients (6) the relations between the stability concepts in the last section are much simpler than in the nonlinear case; for a proof of the following theorem see (Hafstein et al., 2018, Prop. 1).

Theorem 3.1. *The following relations hold for the null solution of system* (6)*:*

i) ASiP is equivalent to GASiP.

ii) p-ES for some $p > 0$ *implies GASiP.*

iii) ASiP/GASiP implies p-ES for all small enough $p > 0$*.*

Unsurprisingly, the GASiP of the null solution of the linearization of the SDE (2) implies the ASiP of the null solution of the original system. Just as for deterministic systems, this can be shown by constructing a global Lyapunov function for the linearized system, which then is a local Lyapunov function for the original nonlinear system. In the stochastic case cf. e.g. (Bjornsson et al., 2018, Th. 3.4), where explicit bounds are given on the area where the Lyapunov function is valid for the original nonlinear system.

Just as quadratic Lyapunov functions $V(\mathbf{x}) = \mathbf{x}^T Q\mathbf{x}$, $Q \succ 0$, are a natural candidate for deterministic, linear, autonomous systems, Lyapunov functions of the algebraic form $V(\mathbf{x}) = \|\mathbf{x}\|_Q^p := (\mathbf{x}^T Q\mathbf{x})^{\frac{p}{2}}$, with $Q \succ 0$ and $p > 0$, are the natural candidate for stochastic, linear, autonomous systems (6). This is justified by the following theorem, where $Q \in \mathbb{R}^{d\times d}$ is an arbitrary symmetric and positive definite matrix.

Theorem 3.2. *The origin is GASiP for the linear system* (6)*, if and only if for all small enough* $p > 0$ *there exist constants* $c_1, c_2, c_3, c_4, c_5 > 0$ *and a (Lyapunov) function* $V \in C(\mathbb{R}^d) \cap C^2(\mathbb{R}^d \setminus \{\mathbf{0}\})$ *such that:*

i) $c_1\|\mathbf{x}\|_Q^p \le V(\mathbf{x}) \le c_2\|\mathbf{x}\|_Q^p$ *for all* $\mathbf{x} \in \mathbb{R}^d$.

ii) $LV(\mathbf{x}) \le -c_3\|\mathbf{x}\|_Q^p$ *for all* $\mathbf{x} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$.

iii) V is positively homogenous of degree p.

iv) $\left|\frac{\partial V}{\partial x_r}(\mathbf{x})\right| \le c_4\|\mathbf{x}\|_Q^{p-1}$ *for all* $r = 1,2,\dots,d$ *and all* $\mathbf{x} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$.

v) $\left|\frac{\partial^2 V}{\partial x_r \partial x_s}(\mathbf{x})\right| \le c_5\|\mathbf{x}\|_Q^{p-2}$ *for all* $r,s = 1,2,\dots,d$ *and all* $\mathbf{x} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$.

Proof. Follows by (Bjornsson et al., 2018, Thms. 3.2, 3.3) and the fact that for all $\mathbf{x} \in \mathbb{R}^d$ we have

$$\lambda_{Q,\min}^{\frac{1}{2}}\|\mathbf{x}\| \le \|\mathbf{x}\|_Q \le \lambda_{Q,\max}^{\frac{1}{2}}\|\mathbf{x}\|,$$

where $\lambda_{Q,\min} > 0$ and $\lambda_{Q,\max} > 0$ are the smallest and largest eigenvalues of Q respectively. □

The function $V(\mathbf{x}) = \|\mathbf{x}\|_Q^p := (\mathbf{x}^T Q\mathbf{x})^{\frac{p}{2}}$ automatically fulfills the condition i) of the last theorem with $c_1 = c_2 = 1$ and with $\alpha > 0$ we have $V(\alpha\mathbf{x}) = \alpha^p V(\mathbf{x})$, i.e. condition iii) is fulfilled. Further, routine calculations give

$$\frac{\partial V}{\partial x_r}(\mathbf{x}) = p\|\mathbf{x}\|_Q^{p-2}\sum_{i=1}^{d} Q_{ir}x_i \qquad (9)$$

and

$$\frac{\partial^2 V}{\partial x_r \partial x_s}(\mathbf{x}) =$$
$$p\|\mathbf{x}\|_Q^{p-4}\left(Q_{sr}\|\mathbf{x}\|_Q^2 + (p-2)\sum_{i,j=1}^{d} Q_{is}Q_{jr}x_i x_j\right),$$

which together with $|x_i| \leq \|\mathbf{x}\| \leq \lambda_{Q,\min}^{-\frac{1}{2}}\|\mathbf{x}\|_Q$ deliver conditions iv) and v) with

$$c_4 = p\lambda_{Q,\min}^{-\frac{1}{2}}\sum_{i=1}^{d}|Q_{ir}|$$

and

$$c_5 = p\left(|Q_{sr}| + |p-2|\lambda_{Q,\min}^{-1}\sum_{i,j=1}^{d}|Q_{is}Q_{jr}|\right).$$

Hence, the only condition for a Lyapunov function remaining for $V(\mathbf{x}) = \|\mathbf{x}\|_Q^p$ is condition ii), i.e. whether $LV(\mathbf{x}) \leq -c_3\|\mathbf{x}\|_Q^p$. Note that this is the case for both the Itô and the Stratonovich interpretation of the solutions, only the generator LV is defined differently.

In (Hafstein et al., 2018) this problem was studied and a linear matrix inequality (LMI) was derived, which verified condition ii) for a given and fixed $Q \succ 0$. This essentially implies that one has to "guess" an appropriate $Q \succ 0$ for a Lyapunov function candidate and then verify if the candidate actually is a Lyapunov function. Even in this quite restrictive setup the authors were able to obtain considerably better results for the interesting example of a damped harmonic oscillator with noise (Khasminskii, 2012, Example 6.6) than previous attempts.

In the next section we will show how the problem of generating a Lyapunov function of the form $V(\mathbf{x}) = \|\mathbf{x}\|_Q^p$ for this system can be formulated as a BMI and how the long and tedious routine computations can be considerably simplified by using the Symbolic Math Toolbox in Matlab. Further, we use the results obtained to verify some of the results obtained in (Hafstein et al., 2018). The important step, however, of solving the BMI, will be dealt with later.

4 BMI FOR THE LYAPUNOV FUNCTION

To derive a bilinear matrix inequality (BMI) feasibility problem for the conditions of a Lyapunov function for an autonomous, linear SDE, we will consider a concrete example, namely, the damped harmonic oscillator

$$\ddot{x} + k\dot{x} + \omega^2 x = 0$$

from (Khasminskii, 2012, Ex. 6.6) and (Hafstein et al., 2018, Ex. 5.1). Setting $x_1 = \omega x$ and $x_2 = \dot{x}$ we get the linear system of equations

$$\dot{\mathbf{x}} = F\mathbf{x}, \text{ with } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ and } F = \begin{pmatrix} 0 & \omega \\ -\omega & -k \end{pmatrix}.$$

The eigenvalues of F are $\lambda_\pm = (-k \pm \sqrt{k^2 - 4\omega^2})/2$ and the origin is globally asymptotically stable for $k > 0$.

We add white noise to the damping and consider the SDE

$$d\mathbf{X}(t) = F\mathbf{X}(t)dt + G^1\mathbf{X}(t)dW_1(t), \tag{10}$$

where

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \text{ and } G^1 = \begin{pmatrix} 0 & 0 \\ 0 & -\sigma \end{pmatrix}.$$

Here W_1 is a one-dimensional Wiener-process and σ quantifies the noise.

As show in (Hafstein et al., 2018, Lemma 4.1) the generator from Definition 2.6 using the Lyapunov functions candidate $V(\mathbf{x}) = \|\mathbf{x}\|_Q^p$ and for the system (6) interpreted in the Itô sense, can be written

$$LV(\mathbf{x}) = -\frac{p}{2}\|\mathbf{x}\|_Q^{p-4}H(\mathbf{x}) \tag{11}$$

with

$$H(\mathbf{x}) = -\mathbf{x}^T\left(F^TQ + QF + \sum_{u=1}^{U}(G^u)^TQG^u\right)\mathbf{x}\|\mathbf{x}\|_Q^2$$
$$+\frac{2-p}{4}\sum_{u=1}^{U}\left(\mathbf{x}^T(QG^u + (G^u)^TQ)\mathbf{x}\right)^2.$$

Similarly, by comparing (7) and (8) and taking formula (9) into account, one sees that by adding the following term on the right-hand-side of (11), one gets the generator (8) for the system (6) interpreted in the Stratonovich sense:

$$\sum_{i,j=1}^{d}\left(\frac{1}{2}\sum_{k=1}^{d}\sum_{u=1}^{U}G_{ij}^uG_{jk}^u x_k\right)\frac{\partial V}{\partial x_i}(\mathbf{x})$$
$$= \|\mathbf{x}\|_Q^{p-2}\frac{p}{2}\sum_{u=1}^{U}\sum_{i,j,k,\ell=1}^{d}G_{ij}^uG_{jk}^u x_k Q_{\ell i}x_\ell$$
$$= \|\mathbf{x}\|_Q^{p-2}\frac{p}{2}\sum_{u=1}^{U}\sum_{j,k,\ell=1}^{d}[QG^u]_{\ell j}G_{jk}^u x_k x_\ell$$
$$= \|\mathbf{x}\|_Q^{p-2}\frac{p}{2}\sum_{u=1}^{U}\sum_{k,\ell=1}^{d}[QG^uG^u]_{\ell k}x_k x_\ell$$
$$= \|\mathbf{x}\|_Q^{p-2}\frac{p}{2}\sum_{u=1}^{U}\mathbf{x}^TQ[G^u]^2\mathbf{x}$$
$$= \|\mathbf{x}\|_Q^{p-2}\frac{p}{4}\sum_{u=1}^{U}\mathbf{x}^T\left(Q[G^u]^2 + [(G^u)^T]^2Q\right)\mathbf{x}$$

In the preceding computations $[QG^u]_{\ell j}$ denotes the (ℓ, j)-entry of the matrix QG^u etc. Alternatively, one can add

$$\widetilde{H}(\mathbf{x}) = -\frac{1}{2}\sum_{u=1}^{U} \mathbf{x}^T \left(Q[G^u]^2 + [(G^u)^T]^2 Q\right)\mathbf{x}\,\|\mathbf{x}\|_Q^2$$

to $H(\mathbf{x})$ in (11), i.e. the generator in the Stratonovich interpretation can be written

$$LV(\mathbf{x}) = -\frac{p}{2}\|\mathbf{x}\|_Q^{p-4}\left(H(\mathbf{x}) + \widetilde{H}(\mathbf{x})\right). \tag{12}$$

Note that both the $H(\mathbf{x})$ and $\widetilde{H}(\mathbf{x})$ are homogenous polynomials in $\mathbf{x} \in \mathbb{R}^d$ of degree 4. In the following we will consider the Itô interpretation, but note that the adaptation to the Stratonovich is straight forward.

The condition ii) from Theorem 3.2 becomes

$$LV(\mathbf{x}) = -\frac{p}{2}\|\mathbf{x}\|_Q^{p-4} H(\mathbf{x}) \le -c_3\|\mathbf{x}\|_Q^p$$

or

$$H(\mathbf{x}) \ge \frac{2c_3}{p}\|\mathbf{x}\|_Q^4. \tag{13}$$

We need to fix the parameter $p > 0$ in $H(\mathbf{x})$ and (13) to proceed. If $c_3 > 0$ is now also fixed we can work further with inequality (13) directly. If, however, we want $c_3 > 0$ to be a parameter in our BMI problem, it is preferable to take advantage of the norm equivalence of $\|\cdot\|$ and $\|\cdot\|_Q$ and consider the stricter inequality

$$H(\mathbf{x}) \ge c\|\mathbf{x}\|^4 \ \text{ with } \ c = \frac{2c_3\lambda_{Q,\max}^2}{p}, \tag{14}$$

that implies (13) and we will do this in the following.

Now we are ready to write the conditions for a Lyapunov function for our ansatz $V(\mathbf{x}) = \|\mathbf{x}\|_Q^p$ for our equation (10) as a BMI. First we fix $p > 0$ and define

$$P_c(x,y) = H(x,y) - c(x^2+y^2)^2.$$

Note that $P_c(x,y) \ge 0$ for all $x, y \in \mathbb{R}$ is equivalent to (14).

The objective is to find a parameter $c > 0$, a symmetric and positive definite matrix $Q \in \mathbb{R}^{2\times 2}$, and a symmetric and positive semi-definite matrix $P \in \mathbb{R}^{3\times 3}$, such that for $Z = (x^2 \ \ xy \ \ y^2)^T$ we have

$$P_c(x,y) = Z^T P Z =: P_Z. \tag{15}$$

More detailed, we want to determine values for the variables

$$c, Q_1, Q_2, Q_3, P_1, P_2, P_3, P_4, P_5, P_6$$

such that $c > 0$,

$$Q = \begin{pmatrix} Q_1 & Q_2 \\ Q_2 & Q_3 \end{pmatrix} \succ 0,$$

$$P = \begin{pmatrix} P_1 & P_2 & P_3 \\ P_2 & P_4 & P_5 \\ P_3 & P_5 & P_6 \end{pmatrix} \succeq 0,$$

and, additionally, such that P_c is the same polynomial, in x and y, as the expression $P_Z := Z^T P Z$. If we succeed in doing this, then $V(\mathbf{x}) = \|\mathbf{x}\|_Q^p$ automatically fulfills all the conditions in Theorem 3.2, except possibly condition ii), just because $Q \succ 0$. Further, the condition ii) is also fulfilled because $P_c = P_Z$, as a polynomial in x and y, is easily seen to be the sum of squared polynomials and thus nonnegative; namely

$$P_c(x,y) = P_Z = \sum_{i=1}^{3}\left(\sqrt{D_i}[OZ]_i\right)^2,$$

where $P = O^T D O$ is the factorization of P using an orthogonal matrix $O \in \mathbb{R}^{3\times 3}$ and diagonal matrix $D \in \mathbb{R}^{3\times 3}$, the $D_i \ge 0$ are the eigenvalues of P, or equivalently the diagonal elements of D, and $[OZ]_i$ is the ith element in the vector OZ, a polynomial in x and y.

To obtain the relations between the variables we have to equate the coefficients of the polynomials P_c and P_Z. This can be done by equating all fourth-order partial derivatives of P_c and P_Z with respect to x and y, i.e.

$$\frac{\partial^4 P_c}{\partial x^4} = \frac{\partial^4 P_Z}{\partial x^4}, \ \frac{\partial^4 P_c}{\partial x^3 \partial y} = \frac{\partial^4 P_Z}{\partial x^3 \partial y}, \ \frac{\partial^4 P_c}{\partial x^2 \partial y^2} = \frac{\partial^4 P_Z}{\partial x^2 \partial y^2}$$

$$\frac{\partial^4 P_c}{\partial x \partial y^3} = \frac{\partial^4 P_Z}{\partial x \partial y^3}, \ \frac{\partial^4 P_c}{\partial y^4} = \frac{\partial^4 P_Z}{\partial y^4}$$

This is most conveniently achieved by using a computer algebra system (CAS). In the following Listing 1 we use Matlab's Symbolic Toolbox to set up the problem as explained above. Note that we keep p as well as the constants $\omega = w, k, \sigma = s$ of our example SDE (10) as parameters:

Listing 1: Setting up the problem.

```
syms Q1 Q2 Q3 P1 P2 P3 P4 P5 P6 c
syms  w k s p
Q=[Q1 Q2;Q2 Q3]
G=[0 0;0 -s]
F=[0 w;-w -k]
syms x y
xv=[x y].'
Pc=-xv.'*(F.'*Q+Q*F+G.'*Q*G)*xv ...
 *xv.'*Q*xv...
 +(2-p)/4*(xv.'*(Q*G+G.'*Q)*xv)^2 ...
 -c*(xv.'*xv)^2
P=[P1 P2 P3; P2 P4 P5; P3 P5 P6]
Z=[x^2 x*y y^2].'
PZ= Z.'*P*Z
```

Obtaining the relations between the variables is now easy using symbolic differentiation, e.g. `diff(PZ,x,4)` gives `24*P1` and `diff(Pc,x,4)` gives `48*Q1*Q2*w - 24*c` and we obtain the relation $P_1 = 2Q_1Q_2\omega - c$ from $\partial^4 P_c/\partial x^4 = \partial^4 P_Z/\partial x^4$. This procedure, however, becomes tedious for obtaining the coefficients for Q_1Q_1, Q_2Q_2, Q_3Q_3, Q_1Q_2, Q_1Q_3, Q_2Q_3, c from P_c ; in terms of the parameters p, ω, k, σ. A more automated process is given in Listing 2 and one can immediately read from its output `diff(P_Z,x,4)=P1` and `Q1*Q1 Q2*Q2 Q3*Q3 Q1*Q2 Q1*Q3 Q2*Q3 c = 0, 0, 0, 2*w, 0, 0, -1` the same information, i.e. $P_1 = 2\omega \cdot Q_1Q_2 + (-1)\cdot c$

Listing 2: Obtaining the relations: $\partial^4/\partial x^4$.

```
eq=diff(PZ,x,4)/factorial(4);
fprintf('\ndiff(P_Z,x,4)=%s\n',char(eq))
eq=diff(Pc,x,4)/factorial(4);
fprintf('diff(P_c,x,4)=%s\n',char(eq))
eq1=diff(eq,Q1,2)/2;
eq2=diff(diff(eq,Q1),Q2);
eq3=diff(eq,Q2,2)/2;
eq4=diff(eq,c);
eq5=diff(eq,Q3,2)/2;
eq6=diff(diff(eq,Q1),Q3);
eq7=diff(diff(eq,Q2),Q3);
fprintf('Q1*Q1  Q2*Q2  Q3*Q3  Q1*Q2  ...
 Q1*Q3  Q2*Q3  c  =  %s,  %s,  %s,  %s,  ...
 %s,  %s,  %s\n\n',char(eq1),  ...
 char(eq3),  char(eq5),  char(eq2),  ...
 char(eq6),  char(eq7),char(eq4))
```

The other cases are similar and can be implemented by adapting the code in Listing 2, e.g. for the case $\partial^4 P_c/\partial x^2\partial y^2 = \partial^4 P_Z/\partial x^2\partial y^2$ just change `eq=diff(PZ,x,4)/factorial(4);` to `eq=diff(diff(PZ,x,2),y,2)/(2*2);` and similarly `eq=diff(Pc,x,4)/factorial(4);` to `eq=diff(diff(Pc,x,2),y,2)/(2*2);`

The results from this procedure are the following relations between the variables:

Variable Relations:

$$\begin{aligned}
P_1 &= 2\omega Q_1Q_2 - c\\
P_2 &= -\omega Q_1^2 + 2\omega Q_2^2 + kQ_1Q_2 + \omega Q_1Q_3\\
2P_3 + P_4 &= (4k - p\sigma^2 + 2\sigma^2)Q_2^2 - 6\omega Q_1Q_2\\
&\quad + (2k-\sigma^2)Q_1Q_3 + 6\omega Q_2Q_3 - 2c\\
P_5 &= -2\omega Q_2^2 + \omega Q_3^2 - \omega Q_1Q_3\\
&\quad + (3k - p\sigma^2 + \sigma^2)Q_2Q_3\\
P_6 &= (2k - p\sigma^2 + \sigma^2)Q_3^2 - 2\omega Q_2Q_3 - c.
\end{aligned}$$

We now come to writing our BMI problem. One formulation of a BMI feasibility problem is to determine values for the variables $q_1, q_2, \dots, q_M$, given symmetric matrices $A_{ij}, B_i, C \in \mathbb{R}^{N\times N}$, $i,j = 1,2,\dots,M$, such that

$$\sum_{i=1}^{M}\sum_{j=i}^{M} q_iq_jA_{ij} + \sum_{i=1}^{M} q_iB_i + C \succeq 0. \tag{16}$$

Our problem has the variables

$$q_1 = Q_1, q_2 = Q_2, q_3 = Q_3, q_4 = P_3, q_5 = P_4, q_6 = c,$$

because the original variables P_1, P_2, P_5, P_6 can be written in terms of $q_1 = Q_1$, $q_2 = Q_2$, $q_3 = Q_3$, $q_6 = c$.

To define the appropriate matrices let us first define the matrices $E_{ii} = \mathbf{e}_i\mathbf{e}_i^T$ and $E_{ij} := \mathbf{e}_i\mathbf{e}_j^T + \mathbf{e}_j\mathbf{e}_i^T$ for $i<j$. Here $\mathbf{e}_i$ is the usual ith unit vector in $\mathbb{R}^8$.

The constraints of our BMI feasibility problem for the SDE (10) can be written

$$\sum_{i=1}^{6}\sum_{j=i}^{6} q_iq_jA_{ij} + \sum_{i=1}^{6} q_iB_i + C \succeq 0 \tag{17}$$

with the following nonzero matrices and where $\varepsilon > 0$ is a small parameter.

Matrix Definitions:

$$\begin{aligned}
A_{11} &= -\omega E_{12},\\
A_{12} &= 2\omega E_{11} + kE_{12} - 6\omega(E_{44} - E_{55})\\
A_{13} &= \omega E_{12} + (2k-\sigma^2)(E_{44} - E_{55}) - \omega E_{23}\\
A_{22} &= 2\omega E_{12} + (4k - p\sigma^2 + 2\sigma^2)(E_{44} - E_{55}) - 2\omega E_{23}\\
A_{23} &= 6\omega(E_{44} - E_{55}) + (3k - p\sigma^2 + \sigma^2)E_{23} - 2\omega E_{33}\\
A_{33} &= \omega E_{23} + (2k - p\sigma^2 + \sigma^2)E_{33}\\
B_1 &= E_{66}\\
B_2 &= E_{67}\\
B_3 &= E_{77}\\
B_4 &= E_{13} + 2(E_{55} - E_{44})\\
B_5 &= E_{22} + E_{55} - E_{44}\\
B_6 &= -E_{11} - 2(E_{44} - E_{55}) - E_{33} + E_{88}\\
C &= -\varepsilon(E_{66} + E_{77} + E_{88})
\end{aligned}$$

All other matrices are set to the zero 8×8 matrix. Note that one can also consider the left-hand-side of (17) as matrix $M \in \mathbb{R}^{8\times 8}$, whose entries are quadratic functions in the variables q_i.

A few comments are in order. First, since the BMI problem (16) is written in terms of semi-definiteness, i.e. $\succeq$ rather than $\succ$, conditions like $c>0$ and $Q \succ 0$ must be written as $c - \varepsilon \geq 0$ and $Q - \varepsilon I \succeq 0$ respectively, for a fixed, small parameter $\varepsilon > 0$.

Second, the constraints $P \succeq 0$ are implemented in the principal submatrix $M_{1:3,1:3} := (M_{ij})_{i,j=1,2,3}$, the constraints $Q \succ 0$ in the principal submatrix $M_{6:7,6:7}$, and the constraints $c > 0$ in the principal submatrix (or element) $M_{8,8}$. The principal submatrices (elements) $M_{4,4}$ and $M_{5,5}$ are used to force the equality

$$2P_3 + P_4 = (4k - p\sigma^2 + 2\sigma^2)Q_2^2 - 6\omega Q_1 Q_2 + (2k - \sigma^2)Q_1 Q_3 + 6\omega Q_2 Q_3 - 2c. \quad (18)$$

Apart from these principal submatrices the entries of M are zero. Lets go through the constraints in more detail:

Principal Submatrix $M_{1:3,1:3}$:
From the Matrix Definitions and (17) we see, by looking for E_{11} in the Matrix Definitions, that

$$M_{1,1} = 2\omega q_1 q_2 - q_6.$$

From the definitions of the variables q_i and the Variable Relations this implies $M_{1,1} = 2\omega Q_1 Q_2 - c = P_1$. In exactly the same way we get $M_{2,1} = M_{1,2} = P_2$, $M_{2,3} = M_{3,2} = P_5$, $M_{3,3} = P_6$, where P_2, P_5, P_6, just as P_1, are written in terms of the $q_1 = Q_1$, $q_2 = Q_2$, $q_3 = Q_3$, $q_6 = c$. The variables $q_4 = P_3$ and $q_5 = P_4$ cannot be written directly in terms of the other variables, and are thus put into $M_{1:3,1:3}$ as $M_{1,3} = M_{3,1} = q_4 = P_3$ and $M_{2,2} = q_5 = P_4$.

Principal Submatrices $M_{4,4}, M_{5,5}, M_{8,8}$:
Since we only have $\geq$ and not $=$ in the BMI feasibility problem (17) at our service, we implement the equality (18) through

$$M_{4,4} = \text{rhs} - \text{lhs} \geq 0 \quad \text{and} \quad M_{5,5} = \text{lhs} - \text{rhs} \geq 0,$$

where "lhs" and "rhs" denote the left-hand-side and the right-hand-side of equation (18), respectively. Further,

$$c = q_6 = M_{8,8} - \varepsilon \geq 0.$$

Principal Submatrix $M_{6:7,6:7}$:
The implementation is obvious:

$$M_{6:7,6:7} = \begin{pmatrix} q_1 - \varepsilon & q_2 \\ q_2 & q_3 - \varepsilon \end{pmatrix} = \begin{pmatrix} Q_1 & Q_2 \\ Q_2 & Q_3 \end{pmatrix} - \varepsilon I \succeq 0.$$

5 VERIFICATION

We verified numerically our construction of the BMI feasibility problem (17) for the SDE (10) for some sets of parameters $k, \omega, \sigma, p, c, Q$ that were used in the sum-of-squared (SOS) approach in (Hafstein et al., 2018, Ex. 5.1). For example, with

$$\omega = 2.75,\ k = 0.9,\ \sigma = 2,\ p = 0.1,\ Q = \begin{pmatrix} 3 & 1/3 \\ 1/3 & 3 \end{pmatrix}$$

one can fix

$$q_1 = 3,\ q_2 = 1/3,\ q_3 = 3,\ q_6 = 0.05 = c,$$

and find values for the variables q_4 and q_5 such that the BMI feasibility problem (17) has a solution; indeed, with $q_4 = -12.126111$ and $q_5 = 5.596667$ the minimum eigenvalue of $M_{1:3,1:3}$ is greater than 0.02.

If ω is changed from $\omega = 2.75$ to $\omega = 2.5$ it is not possible to find values for q_4 and q_5 such that the BMI feasibility problem (17) has a solution; even with $q_6 = c = 0$ the minimum eigenvalue of $M_{1:3,1:3}$ is always negative (less than -0.15), see Figure 1.

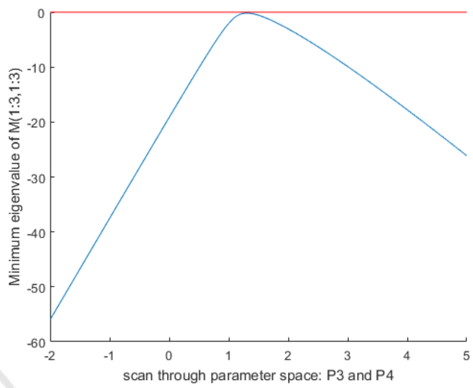

Figure 1: The minimum eigenvalue of $M_{1:3,1:3} = P$ for a scan through values for the variables $q_4 = P_3$ and $q_5 = P_4$. For no admissible values for q_4 and q_5 is the minimum positive.

These results fully conform to the findings obtained in (Hafstein et al., 2018, Ex. 5.1). The code for producing these results is given in Listing 3.

Listing 3: Verification.

```
% make the Eij
E=eye(8);
E11=E(:,1)*E(1,:);
E22=E(:,2)*E(2,:);
E33=E(:,3)*E(3,:);
E44=E(:,4)*E(4,:);
E55=E(:,5)*E(5,:);
E66=E(:,6)*E(6,:);
E77=E(:,7)*E(7,:);
E88=E(:,8)*E(8,:);
E12=E(:,1)*E(2,:)+E(:,2)*E(1,:);
E13=E(:,1)*E(3,:)+E(:,3)*E(1,:);
E23=E(:,2)*E(3,:)+E(:,3)*E(2,:);
E67=E(:,6)*E(7,:)+E(:,7)*E(6,:);

% works
w=2.75; k=0.9; s=2.0; p=0.1; c=0.05;
q1=3.0; q2=1/3.0; q3=3.0; q6=c;
% w=2.5; c=0.05; q6=c; % does not work
V=(4*k-p*s^2+2*s^2)*q2^2-6*w*q1*q2 ...
 +(2*k-s^2)*q1*q3+6*w*q2*q3-2*c;
eps=0.05;
% make A, B, C as possible
A11=-w*E12;
A12=2*w*E11+k*E12-6*w*(E44-E55);
A13=w*E12+(2*k-s^2)*(E44-E55)-w*E23;
```

```
A22=2*w*E12+(4*k-p*s^2+2*s^2) ...
 *(E44-E55)-2*w*E23;
A23=6*w*(E44-E55)+(3*k-p*s^2+s^2) ...
 *E23-2*w*E33;
A33=w*E23+(2*k-p*s^2+s^2)*E33;
B1=E66;B2=E67;B3=E77;
B4=E13+2*(E55-E44);
B5=E22+E55-E44;
B6=-E11-2*(E44-E55)-E33+E88;
C=-eps*(E66+E77+E88);

% 2*P3+P4=V
xmin=-2; xmax=5; steps=1000;
x=zeros(steps,1);
me=zeros(steps,1);
for i=1:steps
  x(i)=xmin+i/steps*(xmax-xmin);
  q4=x(i)*V/2; q5=V-2*q4;
  M=q1^2*A11+q1*q2*A12+q1*q3*A13 ...
   +q2^2*A22+q2*q3*A23+q3^2*A33 ...
   +q1*B1+q2*B2+q3*B3+q4*B4 ...
   +q5*B5+q6*B6+C;
  me(i)=min(eig(M(1:3,1:3)));
end
fprintf('max-min eig = %d\n',max(me))
hold on
plot(x,me)
plot(x,zeros(length(x),1),'r')
```

6 CONCLUSIONS

We presented a bilinear matrix inequality (BMI) approach for the computation of Lyapunov functions for autonomous, linear stochastic differential equations (SDE). For a concrete example we showed how to derive the BMI problem and we verified the results of our approach by comparing with previous findings. Future work will be aimed at writing software to generate the BMI problem automatically for a general autonomous, linear SDE and solving the BMI problem numerically.

ACKNOWLEDGEMENT

The research done for this paper was supported by the Icelandic Research Fund (Rannís) in the project 'Lyapunov Methods and Stochastic Stability' (152429-051), which is gratefully acknowledged.

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