Some Fixed-Point Results on N_b-Cone Metric Spaces

Jerolina Fernandez
Department of Mathematics and Statistics, The Bhopal School of Social Sciences, Bhopal, India

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Abstract: In mathematical analysis, diverse generalizations of metric spaces like 2-Metric, D-metric, G-metric, S-metric, b-metric, Cone metric, and N-cone metric spaces have been studied. Malviya et al. (2012) introduced N-cone metric spaces, a generalization of cone and S-metric spaces, exploring their properties in fixed-point theory. This paper extends and revises results from Wang et al. (1984) in this novel context. Theorems and corollaries demonstrate the uniqueness and existence of fixed points under specified conditions. These findings enrich the understanding of generalized metric spaces and their applications in mathematical analysis.

1 INTRODUCTION

In the literature of Mathematical Analysis there are various generalization of metric spaces like as 2-Metric space ('Gahler 1963, 1966'), D-metric space ('Mustafa Z. and Sims B. [2003, 2006]'), G-metric Space ('Dhage B.C., 1992'), S-metric Space ('Shaban S. et al., 2012'), b-metric Space ('Bakhtin, I.A., 1989'), Cone metric space ('Huang et al., 2007') etc. In 2012, ('Malviya et al. [accepted in FILOMAT]') defined a new structure namely N-cone metric space, which was the generalization of cone metric space and S-metric space, and studied various properties and their applications in fixed point theory. In this paper we extend and modify the results of ('Wang et al., 1984') in this new setting**.**

Definition 1.1 (Gahlers 1963, 1966). Let X be a nonempty set. A generalized metric (or 2-metric) on X is a function d: $X^3 \to R^+$ that satisfies the following conditions for all $x, y, z, a \in X$.
d$(x, y, z) \geq 0$
d$(x, y, z) = 0$ if and only if $x = y = z$
d(x, y, z) = d(p{x, y, z}), (symmetry),where p is permutation function,
d$(x, y, z) \leq$ d(x, y, a) + d(x, a, z) + d(a, y, z) for all $x, y, z, a \in X$.
Then the function d is called a 2- metric and the pair (X, d) is called a 2-metric space.

Definition 1.2 (Mustafa 2003, 2006)**.** Let X be a nonempty set. A G- metric on X is a function $G: X^3 \to [0, \infty)$ that satisfies the following conditions for all $x, y, z, a \in X$.

$$G(x, y, z) = 0 \ if \ x = y = z$$
$$0 < G(x, y, z) \ for \ all \ x, y \in X \ with \ x \neq y,$$
$$G(x, x, y) \leq G(x, y, z) for \ all \ x, y, z \in X \ with \ x \neq y,$$
$$G(x, y, z) = G(x, z, y) = G(y, z, x) \dots \dots \dots.$$
$$G(x, y, z) \leq G(x, a, a) + G(a, y, z) \text{ for all } x, y, z, a \in X.$$

Then the function G is called an G- metric and the pair (X, G) is called a G-metric space.

Definition 1.3 (Sedghi, 2012)**.** Let X be a nonempty set. An S-metric on X is a function $S: X^3 \to [0, \infty)$ that satisfies the following conditions for all $x, y, z, a \in X$.

$$S(x, y, z) \geq 0$$
$$S(x, y, z) = 0 \text{ if and only if } x = y = z$$
$$S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$$

Then the function S is called an S- metric and the pair (X, S) is called an S-metric space

Definition 1.4 (Bakhtin, 1989)**.** Let X be a nonempty set and $s \geq 1$ a given real number. A function $d : X \times X \to R^+$ is a b-metric on X if, for all $x, y, z \in X$, the following conditions hold:
(1) $d(x, y) = 0$ if and only if $x = y$,
(2) $d(x, y) = d(y, x)$,
(3) d$(x, z) \leq s$ [d(x, y) +d (y, z)].

In this case, the pair (X, d) is called a b-metric space

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Definition 1.5 (Huang and Zhang, 2007). Let X be a nonempty set and E be the real Banach space. Suppose the mapping $d{:}X{\times}X \to E$ satisfies

1. $\mathbf{0} \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \mathbf{0}$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 1.6 (Malviya N., Fisher, 2003). Let X be a nonempty set. An N-cone metric on X is a function $N: X^3 \to E$, that satisfies the following conditions for all $x, y, z, a \in X$.

$$N(x, y, z) \geq 0$$
$$N(x, y, z) = 0 \text{ if and only if } x = y = z$$
$$N(x, y, z) \leq N(x, x, a) + N(y, y, a) + N(z, z, a)$$

Then the function N is called an N-cone metric and the pair (X, N) is called an N-cone metric space.

2 MAIN RESULTS

Definition 2.1. Let X be a nonempty set, E is the real Banach space and $s \geq 1$ be a given real number. An N_b-cone metric on X is a function $N_b: X^3 \to E$, that satisfies the following conditions for all $x, y, z, a \in X$.

$$N_b(x, y, z) \geq 0$$
$$N_b(x, y, z) = 0 \text{ if and only if } x = y = z$$
$$N_b(x, y, z) \leq s[N_b(x, x, a) + N_b(y, y, a) + N_b(z, z, a)]$$

Then the function N_b is called an N_b-cone metric and the pair (X, N_b) is called an N_b-cone metric space.

Definition 2.2. If (X, N_b) is an N_b-cone metric space, then it is called symmetric if for all $x, y, z \in X$, we have $N_b(x, x, y) = N_b(y, y, x)$.

Definition 2.3. Let (X, N_b) be an N_b-cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $0 \ll c$ there is N such that for all $n > N, N_b(x_n, x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x and x is the limit of $\{x_n\}$. We denote this by $x_n \to x$ as $(n \to \infty)$.

Lemma 1. Let (X, N_b) be an N_b-cone metric space and P be a normal cone with normal constant k. Let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x and $\{x_n\}$ also convergesto y then $x = y$. That is the limit of $\{x_n\}$, if exists is unique.

Definition 2.4. Let (X, N_b) be an N_b-cone metric space and $\{x_n\}$ be a sequence in X. If for any $c \in E$ with $0 \ll c$ there is N such that for all$m, n > N, N_b(x_n, x_n, x_m) \ll c$, then$\{x_n\}$ is called a Cauchy sequence in X.

Definition 2.5. Let (X, N_b) be an N_b-cone metric space. If every Cauchy sequence in X is convergent in X, then X is called a complete N_b-cone metric space.

Lemma 2. Let (X, N_b) be an N_b-cone metric space and $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x, then $\{x_n\}$ is a Cauchy sequence.

Definition 2.6. Let (X, N_b) and $(X', N_b{}')$ be N_b-cone metric spaces. Then a function $f: X \to X'$is said to be continuous at a point $x \in X$ if and only if it is sequentially continuous at x, that is whenever $\{x_n\}$ is convergent to x we have $\{fx_n\}$ is convergent to$f(x)$.

Lemma 3. Let (X, N_b) be an N_b-cone metric space and P be a normal cone with normal constant k. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X and suppose that $x_n \to x, y_n \to y$ as $n \to \infty$. Then $N_b(x_n, x_n, y_n) \to N_b(x, x, y)$ as $n \to \infty$.

Remark 1. If $x_n \to x$ in an N_b-cone metric space in X then every subsequence of $\{x_n\}$ converges to x in X.

Proposition 1. Let (X, N_b) be an N_b-cone metric space and P be a cone in a real Banach space E. If $u \leq v, v \ll w$ then $u \ll w$.

Lemma 4. Let (X, N_b) be an N_b-cone metric space, P be an N_b-cone in a real Banach space E and $k_1, k_2, k_3, k_4, k > 0$. If $x_n \to x, y_n \to y, z_n \to z$ and $p_n \to p$ in X and

$$ka \leq k_1 N_b(x_n, x_n, x) + k_2 N_b(y_n, y_n, y) + k_3 N_b(z_n, z_n, z) + k_4 N_b(p_n, p_n, p) \ldots \quad (1.1.1)$$

then $a = 0$.

Expansive Map: We define expansive map in N_b-cone metric space as follows

Definition 2.7. Let (X, N_b) be an N_b-cone metric space. A map $f: X \to X$ is said to be an expansive mapping if there exists a constant $L > 1$ such that $N_b(fx, fx, fy) \geq L N_b(x, x, y)$for all $x, y \in X$.

Example 1. Let (X, N_b) be an N_b-cone metric space. Define a self map $f: X \to X$ by $fx = \beta x$ where $\beta > 1$, for all $x \in X$. Clearly f is an expansive map in X.

Theorem 1. Let (X, N_b) be a complete symmetric N_b-cone metric space with respect to a cone P contained in a real Banach space E. Let f and g be two surjective continuous self map of X satisfying.

$$N_b(fx, fx, gy) + k[N_b(x, x, gy) + N_b(y, y, fx)] \geq aN_b(x, x, fx) + bN_b(y, y, gy) + cN_b(x, x, y) \quad (1.1.1)$$

for every $x, y \in X, x \neq y$ where $a, b, c, k \geq 0$, $3sk + 1 < c$. Then f and g have a unique common fixed point in X.

Proof: We define a sequence $\{x_n\}$ as follows for $n = 0,1,2,3, \ldots$

$$x_{2n} = fx_{2n+1}, x_{2n+1} = gx_{2n+2} \quad (1.1.2)$$

If $x_{2n} = x_{2n+1} = x_{2n+2}$ for some n then we see that x_{2n} is a fixed point of f and g. Therefore, we suppose that no two consecutive terms of sequence $\{x_n\}$ are equal.

Now we put $x = x_{2n+1}$ and $y = x_{2n+2}$ in (1.1.1) we get

$$N_b(fx_{n+1}, fx_{2n+1}, gx_{2n+2}) + k[N_b(x_{2n+1}, x_{2n+1}, gx_{2n+2}) + N_b(x_{2n+2}, x_{2n+2}, fx_{2n+1})] \geq aN_b(x_{2n+1}, x_{2n+1}, fx_{2n+1}) + bN_b(x_{2n+2}, x_{2n+2}, gx_{2n+2}) + cN_b(x_{2n+1}, x_{2n+1}, x_{2n+2})$$

$$N_b(x_{2n}, x_{2n}, x_{2n+1}) + k[N_b(x_{2n+1}, x_{2n+1}, x_{2n+1}) + N_b(x_{2n+2}, x_{2n+2}, x_{2n})] \geq aN_b(x_{2n+1}, x_{2n+1}, x_{2n}) + bN_b(x_{2n+2}, x_{2n+2}, x_{2n+1}) + cN_b(x_{2n+1}, x_{2n+1}, x_{2n+2})$$

$$\Rightarrow N_b(x_{2n}, x_{2n}, x_{2n+1}) + k[2sN_b(x_{2n+2}, x_{2n+2}, x_{2n+1}) + sN_b(x_{2n}, x_{2n}, x_{2n+1})] \geq aN_b(x_{2n+1}, x_{2n+1}, x_{2n}) + bN_b(x_{2n+2}, x_{2n+2}, x_{2n+1}) + cN_b(x_{2n+1}, x_{2n+1}, x_{2n+2})$$

$$\Rightarrow (1 + sk - a)N_b(x_{2n}, x_{2n}, x_{2n+1}) \geq (b + c - 2sk)N_b(x_{2n+1}, x_{2n+1}, x_{2n+2})$$

$$\Rightarrow N_b(x_{2n+1}, x_{2n+1}, x_{2n+2}) \leq \frac{(1 + sk - a)}{(b + c - 2sk)} N_b(x_{2n}, x_{2n}, x_{2n+1})$$

$$\Rightarrow N_b(x_{2n+1}, x_{2n+1}, x_{2n+2}) \leq K_1 N_b(x_{2n}, x_{2n}, x_{2n+1})$$

where $K_1 = \frac{(1+sk-a)}{(b+c-2sk)} < 1$ $(As\ \ a + b + c > 1 + 3sk)$

Similarly, we can calculate

$$\Rightarrow N_b(x_{2n+2}, x_{2n+2}, x_{2n+3}) \leq K_2 N_b(x_{2n+1}, x_{2n+1}, x_{2n+2})$$

Where $K_2 = \frac{(1+sk-a)}{(b+c-2sk)} < 1$ $(As\ \ a + b + c > 1 + 3sk)$ and so on.

$\Rightarrow N_b(x_n, x_n, x_{n+1}) \leq KN_b(x_{n-1}, x_{n-1}, x_n)$ for $n = 1,2,3, \ldots$

where $K = max\{K_1, K_2\}$ then $K < 1$.

$$\Rightarrow N_b(x_n, x_n, x_{n+1}) \leq K^n N_b(x_0, x_0, x_1)$$

Now we shall prove that $\{x_n\}$ is a Cauchy sequence. For this for every positive integer p, we have

$$\begin{aligned} N_b(x_n, x_n, x_{n+p}) &\leq 2sN_b(x_n, x_n, x_{n+1}) + 2s^2N_b(x_{n+1}, x_{n+1}, x_{n+2}) + \cdots + 2s^{n+p-1}N_b(x_{n+p-2}, x_{n+p-2}, x_{n+p-1}) + s^{n+p}N_b(x_{n+p-1}, x_{n+p-1}, x_{n+p}) \\ &\leq 2sN_b(x_n, x_n, x_{n+1}) + 2s^2N_b(x_{n+1}, x_{n+1}, x_{n+2}) + \cdots + 2s^{n+p-1}N_b(x_{n+p-2}, x_{n+p-2}, x_{n+p-1}) + 2s^{n+p}N_b(x_{n+p-1}, x_{n+p-1}, x_{n+p}) \end{aligned}$$

$$\begin{aligned} &\leq [2sK^n + 2s^2K^{n+1} + \cdots + 2s^{n+p-1}K^{n+p-2} + 2s^{n+p}K^{n+p-1}]N_b(x_0, x_0, x_1) \\ &= 2sK^n[1 + sK + s^2K^2 + \cdots]N_b(x_0, x_0, x_1) \\ &< \frac{2sK^n}{(1-sK)} N_b(x_0, x_0, x_1) \end{aligned}$$

$$\Rightarrow \|N_b(x_n, x_n, x_{n+p})\| < \frac{2sK^n}{(1 - sK)} K \|N_b(x_0, x_0, x_1)\|$$

which implies that $\|N_b(x_n, x_n, x_{n+p})\| \to 0$ as $n \to \infty$.

Since $K \to 0$ as $n \to \infty$.

Therefore $\{x_n\}$ is a Cauchy sequence in X, which is complete space, so $\{x_n\} \to x \in X$.

Existence of Fixed Point: Since mappings are continuous therefore existence of fixed point follows very easily. As shown below

$$x = x_{2n} = fx_{2n+1} = x_{2n+1} = fx (as\ \ n \to \infty\ \{x_{2n+1}\} \to x)$$

Similarly

$$x = x_{2n+1} = gx_{2n+2} = x_{2n+2} = gx\ (as\ \ n \to \infty\ \{x_{2n+2}\} \to x)$$

$$x = y \quad (1.1.3)$$

which shows that x is a common fixed point of f and g.

Uniqueness: Let z be another common fixed point of f and g, that is

$$fz = z\ and\ gz = z \quad (1.1.4)$$

$$\begin{aligned} N_b(x, x, z) &= N_b(fx, fx, gz) \\ &\geq -k[N_b(x, x, gz) + N_b(z, z, fx)] + aN_b(x, x, fx) + bN_b(z, z, gz) + cN_b(x, x, z) \end{aligned}$$

$$\Rightarrow N_b(x, x, z) \geq -k[N_b(x, x, z) + N_b(z, z, x)] + aN_b(x, x, x) + bN_b(z, z, z) + cN_b(x, x, z)[by\ 1.1.4]$$

$$\Rightarrow (1 - c + 2k)N_b(x, x, z) \geq 0$$

$\Rightarrow N_b(x,x,z) = 0 \qquad [As\ c > 3sk + 1 > 2k + 1]$
$\Rightarrow x = z$
This completes the proof of the Theorem 1.

Corollary 2. Let (X, N_b) be a complete symmetric N_b-cone metric space with respect to a cone P contained in a real Banach space E. Let f and g be two surjective continuous self maps of X satisfying.

$$N_b(fx, fx, gy) \geq cN_b(x, x, y)$$

where $c > 1$. Then f and g have a unique common fixed point in X.

Proof: If we put $k, a, b = 0$ in Theorem 1, then we get above Corollary 2.

Corollary 3. Let (X, N_b) be a complete symmetric N_b-cone metric space with respect to a cone P contained in a real Banach space E. Let f be a continuous surjective self map of X satisfying.

$$N_b(fx, fx, fy) \geq cN_b(x, x, y) \qquad (1.1.5)$$

where $c > 1$. Then f has a unique fixed point in X.

Proof: If we put $f = g$ in Corollary 2 then we get above Corollary 3 which is an extension of Theorem 1 of Wang et al. (Wang et al., 1984) in N_b-cone metric space.

Corollary 4. Let (X, N_b) be a complete symmetric N_b-cone metric space and $f: X \to X$ be a continuous surjection. Suppose that there exists a positive integer n and a real number $C > 1$ such that $N_b(f^n x, f^n x, f^n y) \geq CN_b(x, x, y)$ for all $x, y \in X$. Then f has a unique fixed point in X.

Proof: From Corollary 3, f^n has a unique fixed point z. But $f^n(fz) = f(f^n z) = fz$, so fz is also a fixed point of f^n. Hence $fz = z$, z is a fixed point of f. Since the fixed point of f is also fixed point of f^n, the fixed point of f is unique.

Corollary 5. Let (X, N_b) be a complete symmetric N_b-cone metric space with respect to a cone Pcontained in a real Banach space E. Let f and g be two continuous surjective self maps of X satisfying.

$$N_b(fx, fx, gy) \geq aN_b(x, x, fx) + bN_b(y, y, gy) + cN_b(x, x, y)$$

for every $x, y \in X, x \neq y$ where $a, b, c \geq 0$ and $c > 1$. Then f and g have a unique common fixed point in X.

Proof: The proof is similar to proof of the Theorem 1.

Corollary 6. Let (X, N_b) be a complete symmetric N_b-cone metric space with respect to a cone P contained in a real Banach space E. Let f be surjective continuous self map of X satisfying.

$$N_b(fx, fx, fy) \geq aN_b(x, x, fx) + bN_b(y, y, fy) + cN_b(x, x, y)$$

for every $x, y \in X, x \neq y$ where $a, b, c \geq 0$ and $c > 1$. Then f has a unique fixed point in X.

Proof: If we put $f = g$ in Corollary 5 then we get above Corollary 6 which is an extension of Theorem 2 of Wang et al. [10] in N_b-cone metric space.
The following example demonstrates Corollary 3.

Example 2. Let $E = R^3, P = \{(x, y, z) \in E, x, y, z \geq 0\}$ and $X = R$ and $N_b: X \times X \times X \to E$ is defined by

$$N_b(x, y, z) = (\alpha(|x - z| + |y - z|)^2, \beta(|x - z| + |y - z|)^2, \gamma(|x - z|\ |y - z|)^2)$$

where α, β, γ are positive constants. Then (X, N_b) is a symmetric N_b-cone metric space. Define a self map f on X as follows $fx = 2x$ for all $x \in X$. Clearly f is an expansive mapping. If we take $c = 8$ then condition (1.1.5) holds trivially good and 0 is the unique fixed point of the map f.

Remark 2. In Corollary 6, we proved the fixed point is unique by using only $c > 1$ and there is no need of $a < 1$ and $b < 1$, so it extend and unify the Theorem 2 of Wang et al. [10] in N_b-cone metric space.

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