Pál Type Interpolation Problems with Additional Value Nodes

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Abstract: The author termed Pál type interpolation problems as $PTIP$. In this paper the regularity of $(0,1)-PTIP$ and $(0,2)-PTIP$, with addition of two non-zero complex nodes $\pm\zeta$ or two real nodes ± 1 at value nodes for pairs of considered polynomials is evaluated.

1 INTRODUCTION

L. G. Pál 1975, introduced a new kind of Interpolation on zeros of two different Polynomials. It involves of finding a polynomial of degree $(m+n-1)$, that has prescribed values at m pairwise distinct nodes and prescribed values for r^{th} derivative at n pairwise distinct nodes. These nodes are called value nodes and derivative nodes respectively.

Let π_n be the set of polynomials of degree less than or equal to n with complex coefficients. Let $A(z) \in \pi_n$ and $B(z) \in \pi_m$, then for a given positive integer r the problem of $(0,r)-PTIP$ on the pair $\{A(z), B(z)\}$, is to determine a polynomial $P(z) \in \pi_{n+m-1}$, which assumes arbitrary prescribed values at the zeros of $A(z)$ and arbitrary prescribed values of the r^{th} derivative at the zeros of $B(z)$. The problem is regular if and only if any $P(z)$ satisfying

$$P(y_i) = 0;\text{ where } A(y_i) = 0\ ;\ i = 1,2,\dots,n,$$

$$P^{(r)}(z_j) = 0;\text{ where } B(z_j) = 0\ ;\ j = 1,2,\dots,m,$$

vanishes identically. Here the zeros of $A(z), B(z)$ are assumed to be simple.

(De Bruin and Sharma 2003) observed regularity of $(0, m_1, \dots, m_q)-PTIP$ on the zeros of $(z^n-\alpha_0^n)$, $(z^n-\alpha_1^n), \dots, (z^n-\alpha_q^n)$ with $0<\alpha_0<\alpha_1<,\dots,<\alpha_q$.

(De Bruin 2005) explored necessary and sufficient condition for regularity of $(0,r)-PTIP$ with respect to exchanging value-nodes and derivative-nodes.

(De Bruin and Dikshit 2005) examined regularity of $(0,r)-PTIP$ on the pair $\{(z^m-1)(z-\zeta),(z^n-1)\}$, where m and n are given positive integers and ζ is not a zero of the polynomial (z^m-1). They determined largest domain for ζ, which ensures regularity of the problem. They observed that $(0,r)-PTIP$ on the pair $\{(z^m-1)(z-\zeta),\ (z^n-1)\}$, for positive integers m and n are not regular, if $r>m+1$. For the case, $r \le m+1$ and on the basis of relationship between the positive integers m and n, they explored $(0,r)-$ on some different pairs and found those problems are regular under certain conditions.

(Dikshit 2003) considered $PTIP$ involving Möbius transform of zeros of (z^n+1) and (z^n-1) with one or two extra derivative nodes.

(De Bruin 2005) investigated regularity of $(0,m)-PTIP$ on zeros of the pair $\{w_{n+m}^{(\alpha)}(z), w_n^{(\alpha)}(z)\}$, where α be a complex number with $\alpha^2, \alpha^m, \alpha^n, \alpha^{n+m} \neq 1\ ; n,m \geq 1$.

The method of considering non-uniformly distributed nodes on unit disk is generalized, by involving the Möbius transform of zeros of $(z^{2n}-\rho^{2n})$ on the circle $|z|=\rho'$ (Mandoli and Pathak 2008).

$(0,1)-PTIP$ are found to be regular for following pairs, where $a_m(z) \in \pi_m$ and $b_n(z) \in \pi_n$ with simple zeros, $A_m(z)$ and $B_n(z)$ are the sets of zeros of the polynomials $a_m(z)$ and $b_n(z)$ respectively such that $B_n(z) \subseteq A_m(z)$ (Modi et al 2012)

- $\{a_m(z), (z-\zeta)b_n(z)\}$.
- $\{(z-\zeta)a_m(z), b_n(z)\}$.
- $\{a_m(z), (z-\zeta_1)(z-\zeta_2)b_n(z)\}\ ; \zeta_1 \neq \zeta_2$.

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- $\left\{a_m(z), \prod_{i=1}^{\beta} \quad (z-\zeta_i)b_n(z)\right\}$; ζ_i are pairwise distinct.
- $\{a_m(z), \Psi(t)b_n(z)\}$; $\Psi(t) \in \pi_t (t \geq 2)$ be a polynomial with simple zeros.
- $\{(z-\zeta_1)a_m(z), (z-\zeta_2)b_n(z)\}$.

The author (Pathak and Tiwari 2019, Pathak and Tiwari 2018) revisited regularity of Pál type Birkhoff interpolation and have introduced a new class of $PTIP$. Also, the author (Pathak and Tiwari 2018, Pathak and Tiwari 2020) examined the regularity of 'incomplete' type $PTIP$ on non-uniformly distributed nodes by omitting real and complex nodes and studied 'Incomplete' type $PTIP$ on zeros of polynomials with complex coefficients.

2 MAIN RESULTS

The author considered the polynomials $a_m(z) \in \pi_m$ and $b_n(z) \in \pi_n$ with simple zeros. $A_m(z)$ and $B_n(z)$ are the sets of zeros of the polynomials $a_m(z)$ and $b_n(z)$ respectively such that $B_n(z) \subseteq A_m(z)$. Section 2.1 deals with $(0,1)-PTIP$, while section 2.2 deals with $(0,2)-PTIP$.

$(0,1)-PTIP$ **with two Additional Value Nodes**

Theorem 2.1: *Let* $m,n \geq 1$, *then* $(0,1)-PTIP$ *on* $\{(z^2-\zeta^2)a_m(z), b_n(z)\}$; $\pm\zeta \notin A_m(z)$, $B_n(z) \subseteq A_m(z)$ *is regular.*

Proof: Here, we have total $(m+n+2)$ interpolation points.
We need to determine a polynomial $P(z) \in \pi_{m+n+1}$ with

$$P(y_i) = 0\,;\, y_i \in A_m(z)\,;\, i = 1,2,\dots,m,$$

$$P(\pm\zeta) = 0\,;\; \pm\zeta \notin A_m(z),$$

$$P'(z_j) = 0\,;\, z_j \in B_n(z)\,;\, j = 1,2,\dots,n.$$

Let $P(z) = (z^2-\zeta^2)a_m(z)Q(z)$; where $Q(z) \in \pi_{n-1}$.

Thus $P(z) \in \pi_{m+n+1}$.

The posed problem will be regular, if $P(z) \equiv 0$.

Since $P'(z_j) = 0$, we get

$$(z_j^2-\zeta^2)a_m(z_j)Q'(z_j) + Q(z_j)\left[(z_j^2-\zeta^2)a'_m(z_j) + 2z_j a_m(z_j)\right] = 0.$$

As $z_j \in B_n(z) \subseteq A_m(z)$, we have

$$(z_j^2-\zeta^2)a'_m(z_j)Q(z_j) = 0.$$

Since $\pm\zeta \notin A_m(z)$ and $a_m(z)$ has simple zeros, the polynomial and its derivative cannot vanish simultaneously at the same point, we have

$$Q(z_j) = 0.$$

Since z_j has n values, we get

$$Q(z) = Cq_n(z) \qquad (2.1)$$

According to our assumption $Q(z) \in \pi_{n-1}$ and therefore on account of equation (2.1), we get

$$Q(z) \equiv 0.$$

Corollary 2.1: *Let* $m,n \geq 1$, *then* $(0,1)-PTIP$ on $\{(z^2-1)a_m(z), b_n(z)\}$; $\pm 1 \notin A_m(z)$, $B_n(z) \subseteq A_m(z)$ *is regular.*

$(0,2)-PTIP$ **with two Additional Value Nodes**

Theorem 2.2: *Let* $m,n \geq 1$, *then* $(0,2)-PTIP$ *on* $\{(z^2-\zeta^2)a_m(z), b_n(z)\}$; $\pm\zeta \notin A_m(z)$, $B_n(z) \subseteq A_m(z)$ *is regular.*

Proof: Here, we have total $(m+n+2)$ interpolation points.

We need to determine a polynomial $P(z) \in \pi_{m+n+1}$ with

$$P(y_i) = 0\,;\, y_i \in A_m(z)\,;\, i = 1,2,\dots,m,$$

$$P(\pm\zeta) = 0\,;\, \pm\zeta \notin A_m(z),$$

$$P''(z_j) = 0\,;\, z_j \in B_n(z)\,;\, j = 1,2,\dots,n.$$

Let $P(z) = (z^2-\zeta^2)a_m(z)Q(z)$; where $Q(z) \in \pi_{n-1}$.

Thus $P(z) \in \pi_{m+n+1}$.

The posed problem will be regular, if $P(z) \equiv 0$.

Since $P''(z_j) = 0$, we get

$$\begin{aligned}(z_j^2-\zeta^2)a_m(z_j)Q''(z_j) &\\ + 2\left[(z_j^2-\zeta^2)a'_m(z_j) + 2z_j a_m(z_j)\right]Q'(z_j) &\\ + \left[(z_j^2-\zeta^2)a''_m(z_j) + 4z_j a'_m(z_j) + 2a_m(z_j)\right]Q(z_j) &= 0.\end{aligned}$$

As $z_j \in B_n(z) \subseteq A_m(z)$ and $a_m(z)$ has simple zero, the polynomial and its derivative cannot vanish simultaneously at the same point, we have

$$2\left(z_j^2-\zeta^2\right)a_m'\left(z_j\right)Q'\left(z_j\right) + \left\{\left(z_j^2-\zeta^2\right)a_m''\left(z_j\right) + 4z_j a_m'\left(z_j\right)\right\}Q\left(z_j\right) = 0.$$

Since $Q(z) \in \pi_{n-1}$ and z_j has n values, the differential equation is given by

$$2(z^2-\zeta^2)a_m'(z)Q'(z) + \{(z^2-\zeta^2)a_m''(z) + 4za_m'(z)\}Q(z) = C_1 b_n(z),$$

$$Q'(z) + \left\{\frac{1}{2}\frac{a_m''(z)}{a_m'(z)} + \frac{2z}{(z^2-\zeta^2)}\right\}Q(z) = C\frac{b_n(z)}{(z^2-\zeta^2)a_m'(z)}, \quad (2.2)$$

for some constant $C = \frac{C_1}{2}$.

Integrating factor of differential equation (2.2) is given by

$$\varphi(z) = exp\int\left\{\frac{1}{2}\frac{a_m''(z)}{a_m'(z)} + \frac{2z}{(z^2-\zeta^2)}\right\}dz,$$

$$\varphi(z) = (z^2-\zeta^2)\{a_m'(z)\}^{\frac{1}{2}}.$$

Set $\eta(z) = \{a_m'(z)\}^{\frac{1}{2}}$.

Solution of differential equation (2.2) is given by

$$\varphi(z)Q(z) = C\int\frac{\varphi(t)b_n(t)}{(t^2-\zeta^2)a_m'(t)}dt,$$

$$(z^2-\zeta^2)\,\eta(z)Q(z) = C\int\frac{(t^2-\zeta^2)\eta(t)b_n(t)}{(t^2-\zeta^2)a_m'(t)}dt,$$

$$(z^2-\zeta^2)\,\eta(z)Q(z) = C\int\frac{\eta(t)b_n(t)}{a_m'(t)}dt,$$

$$C\int\frac{\eta(t)b_n(t)}{a_m'(t)}dt = 0 \Rightarrow C = 0.$$

Hence,

$$Q(z) \equiv 0.$$

Corollary 2.2: *Let* $m, n \geq 1$, *then* $(0,2) - PTIP$ *on* $\{(z^2-1)a_m(z), b_n(z)\}$; $\pm 1 \notin A_m(z)$, $B_n(z) \subseteq A_m(z)$ *is regular.*

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