Some Fixed-Point Results of α-Admissible Mappings in Partial Cone Metric Spaces over Banach Algebra

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Abstract: The article presents α-admissible mappings in Partial Cone Metric Spaces (PCMS) over Banach Algebra (BA), exploring fixed-point results in this context. It builds upon the foundational work of Liu and Xu [2013], which introduced CMS over BA, marking a significant step in fixed-point theory. Fernandez et al. [2016] extended this to PCMS over BA, examining fixed-point results for generalized Lipschitz mappings. This study defines and explores α-admissible mappings in the newly established space, offering results that extend previous findings. The main theorem establishes conditions for the existence of fixed-points for α-admissible, generalized Lipschitz self-maps in θ-complete PCMS over BA, contributing to this evolving area of mathematics.

1 INTRODUCTION

Liu and Xu (2013) introduced the notion of CMS over BA by replacing the Banach space by Banach algebra which became a milestone in the study of fixed-point theory. Moreover, they gave some examples to elucidate that fixed-point results in CMS over BA are not equivalent to metric spaces (in usual sense). Recently, Fernandez et al., (2016) introduced the concept of PCMS over BA and studied some fixed-point results for generalized Lipschitz mappings.

Inspired by the previous notion, in this paper we establish some fixed-point results of α-admissible mappings in the newly defined space. Our results generalize and extend the recent result of Malhotra et al. (2015).

2 PRELIMINARIES

First, we define PCMS over BA.

Definition 2.1.[1] A partial cone metric on a nonempty set M is a function $p{:}M\times M\rightarrow A$ such that for all $\beta,\gamma,\delta\in M$:

(p_1) $\beta=\gamma \Leftrightarrow p(\beta,\beta)=p(\beta,\gamma)=p(\gamma,\gamma)$,

(p_2) $\theta \preccurlyeq p(\beta,\beta) \preccurlyeq p(\beta,\gamma)$,

(p_3) $p(\beta,\gamma)=p(\gamma,\beta)$,

(p_4) $p(\beta,\gamma) \preccurlyeq p(\beta,\delta)+p(\delta,\gamma)-p(\delta,\delta)$.

The pair (M, p) is called a PCMS over BA.

Lemma 2.2. ([5]). Let A be a Banach algebra with a unit e, $k\in A$, then $\lim_{n\to\infty}\|k^n\|^{\frac{1}{n}}$ exists and the spectral radius $\rho(k)$ satisfies

$$\rho(k)=\lim_{n\to\infty}\|k^n\|^{\frac{1}{n}}=\inf\|k^n\|^{\frac{1}{n}}.$$

If $\rho(k) <|\ \ |$, then $(\lambda e-k)$ is invertible in A, moreover,

$$(e-k)^{-1}=\sum_{i=0}\ \ \frac{k^i}{\ \ i+1}$$

where λ is a complex constant.

Lemma 2.3. ([2]). If E is a real Banach space with a solid cone P and $\{u_n\}\subset$ P be a sequence with $\|u_n\|\rightarrow 0$ ($n\rightarrow\infty$), then $\{u_n\}$ is a c-sequence.

Lemma 2.4. ([2]). Let A be a Banach algebra with a unit e and $k\in A$. If λ is a complex constant and $\rho(k) <|\ \ |$, then

$$\rho((e-k)^{-1})\frac{1}{|\ \ |-(k)}.$$

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3 DISCUSSION AND MAIN RESULTS

In this section, we introduce the concept of α-admissible mappings in PCMS over BA.

Definition 3.1. Let M be a nonempty set and α: M ×M →[0;∞) be a function. We say that T is α-admissible if $(\beta, \gamma) \in$M, $\alpha(\beta, \gamma) \geq 1 \Rightarrow \alpha(T\beta, T\gamma) \geq 1$.

Example 3.2. Let M = [0, ∞) and A be the set of all real valued function on M which also have continuous derivatives on M with the norm $\|\beta\| = \|\beta\| + \|\beta'\|$. Define multiplication in the usual way. Let $P = \{\beta \in A: \beta(t) \geq 0, t \in M\}$. It is clear that P is a nonnormal cone and A is a Banach algebra with a unit $e = 1$. Define a mapping p: $M \times M \rightarrow A$ by

$$p(\beta, \gamma) = \{\beta e^t \quad , \quad \beta = \gamma \; (\beta + y)e^t \quad , \; otherwise$$

Define a self-map T on M as follows

$$T\beta = \begin{cases} ln\left(1+\frac{\beta}{3}\right), & \beta \in [0,1] \\ \frac{\beta}{2}, & otherwise \end{cases}$$

and $\alpha(\beta, \gamma) = \begin{cases} 1 & ,\beta,\gamma \in [0,1] \\ 0, & otherwise \end{cases}$

Since $ln(1+t) \leq t$ for each t $\in$ [0, 1), for all β, $\gamma \in X$, we have

when $\beta \neq \gamma$

$$p(T\beta, T\gamma)(t) = \left(ln\, ln\left(1+\frac{\beta}{3}\right) + ln\, ln\left(1+\frac{\gamma}{3}\right)\right)e^t$$

$$\preccurlyeq \left(\frac{\beta}{3} + \frac{\gamma}{3}\right)e^t$$

$$= \frac{1}{3}(\beta + \gamma)e^t$$

$$\preccurlyeq \frac{1}{3}p(\beta, \gamma)(t)$$

and when $\beta = \gamma$,

$$p(T\beta, T\beta)(t) = \left(ln\, ln\left(1+\frac{\beta}{3}\right)\right)e^t$$

$$\preccurlyeq \left(\frac{\beta}{3}\right)e^t$$

$$= \frac{1}{3}\beta e^t$$

$$\preccurlyeq \frac{1}{3}p(\beta, \beta)(t).$$

Therefore, $p(T\beta, T\gamma)(t) \preccurlyeq \frac{1}{3}p(\beta, \gamma)(t)$. Thus, T is a Generalized Lipschitz map in M where $\rho(k) = \frac{1}{3} < 1$ and $\alpha(\beta, \gamma) \geq 1$.

Theorem 3.3. Let (M, p) be a θ-complete PCMS over BA Suppose T be a generalized Lipschitz self-map with Lipschitz vector k satisfying:

(i) T is α-admissible;

(ii) there exists $\beta_0 \in$ X such that $\alpha(\beta_0, T\beta_0) \geq 1$;

(iii) T is continuous.

Then T has a fixed-point $\beta^* \in$ M.

Proof. Let $\beta_0 \in$ M such that $\alpha(\beta_0, T\beta_0) \geq 1$. Define a sequence $\{\beta_n\}$ in M such that $\beta_n = T\beta_{n-1}$ ∀ n ∈ N. If $\beta_n = \beta_{n+1}$ ∀ n ∈ N, then $\beta^* = \beta_n$ is a fixed-point for T. Suppose $\beta_n = \beta_{n+1}$ for all n ∈ N. Since T is α-admissible we deduce

$$\alpha(\beta_0, \beta_1) = \alpha(\beta_0, T\beta_0) \geq 1 \Rightarrow \alpha(T\beta_0, T^2\beta_0) = \alpha(\beta_1, \beta_2) \geq 1:$$

Continuing, we get

$$\alpha(\beta_n, \beta_{n+1}) \geq 1 \text{ for all n} \in \text{N}.$$

Now,

$$p(\beta_n, \beta_{n+1}) = p(T\beta_{n-1}, T\beta_n)$$

$$\preccurlyeq k\, p(\beta_{n-1}, \beta_n)$$

$$\preccurlyeq k^n p(\beta_0, \beta_1).$$

For n < m we have

$$p(\beta_n, \beta_m) \preccurlyeq p(\beta_n, \beta_{n+1}) + p(\beta_{n+1}, \beta_m) - p(\beta_{n+1}, \beta_{n+1})$$

$$\preccurlyeq p(\beta_n, \beta_{n+1}) + p(\beta_{n+1}, \beta_m)$$

$$\preccurlyeq p(\beta_n, \beta_{n+1}) + p(\beta_{n+1}, \beta_{n+2}) + p(\beta_{n+2}, \beta_m) - p(\beta_{n+2}, \beta_{n+2})$$

….

$$\preccurlyeq p(\beta_n, \beta_{n+1}) + p(\beta_{n+1}, \beta_{n+2}) + p(\beta_{n+2}, \beta_m)$$

$$\preccurlyeq p(\beta_n, \beta_{n+1}) + p(\beta_{n+1}, \beta_{n+2}) + p(\beta_{n+2}, \beta_{n+3}) + \ldots\ldots + p(\beta_{m-2}, \beta_{m-1}) + p(\beta_{m-1}, \beta_m)$$

$$\preccurlyeq k^n p(\beta_0, \beta_1) + k^{n+1}p(\beta_0, \beta_1) + k^{n+2}p(\beta_0, \beta_1) + \ldots\ldots + k^{m-1}p(\beta_0, \beta_1)$$

$$= k^n[e + k + k^2 + \ldots\ldots\ldots + (k)^{m-n-1}]p(\beta_0, \beta_1)$$

$$\preccurlyeq k^n(e-k)^{-1}p(\beta_0, \beta_1).$$

Then $\|k^n p(\beta_0 ,\beta_1)\|\|\|k^n\|\|\|p(\beta_0 ,\beta_1)\| \rightarrow 0$ (n $\rightarrow \infty$), by Lemma 2.3, $\{k^n p(\beta_0 ,\beta_1)\}$ is a c-sequence. By Lemma 2.2 and Lemma 2.4, $\{\beta_n\}$ is a θ-Cauchy sequence. Since M is complete, $\exists\ \beta^* \in$ M such that $\beta_n \rightarrow \beta^*$as n $\rightarrow \infty$. Therefore,

$$p\ (\beta_n\ , \beta^*) = p\ (\beta_n\ , \beta_m) = p(\beta^*, \beta^*) = \theta\ .$$

Since T is continuous, we have β_{n+1} = T$\beta_n \rightarrow$ Tβ^* as n $\rightarrow \infty$. By uniqueness, β^*= Tβ^*, that is β^* is a fixed-point of T.

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