# Time Fractional Radon Diffusion Equation and Crank Nicholson Finite Difference Algorithm

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Abstract: The motivation behind this paper is to study the spread of radioactive active substance Radon in the air medium. The Radon diffusion equation in air medium has been solved by applying finite difference scheme. Basic Time fractional Radon diffusion equation has been solved using Crank Nicholson method of finite difference scheme. The numerical solution is discussed for its stability and convergence. The stability of obtained solution is significantly validated for the accuracy and consistency of the solution using Mathematica. The Crank Nicholson Finite Difference Scheme is found to be the best suitable scheme after observing the estimates and errors of radon concentration and its graphical representation. The solution to the Radon diffusion equation is observed to be categorically stable and convergent.

## 1 INTRODUCTION

The detail study of Radon Diffusion equation in various mediums serves the motive of study of propagation of Radon gas and its ill effect in our surroundings and its natural growth which is harmful for living beings. Hence the sole purpose of this paper is to detect the Radon concentration in the air medium, by solving the Radon Diffusion Equation by using FDM.

The hypothesis of this research paper hence are:

1. To study the estimates of Radon concentration through air medium
2. To solve the Time Fractional Radon Diffusion equation
3. To apply the Crank Nicholson Finite Difference Scheme to solve the TFRDE
4. To understand the stability of the solution obtained by the FDM.
5. To observe the convergence of solution obtained by computational method
6. To validate the solution by graphical representation of solution using Mathematica.

Crank Nicholson Finite Difference Scheme has been used to solve the time fractional diffusion equation here. The computations have been supported from the numerical data for different parameters from the (Millar, 1993). The structure of this paper has section wise development. Section one includes overview; section II contains the Crank Nicholson FDM and the solution to TFRDE. The section III discusses stability criteria of solution. Section IV includes convergence of the solution. Section V includes conclusion and discussions.

## 2 CRANK NICHOLSON FINITE DIFFERENCE ALGORITHM

The finite difference algorithms try to solve Fractional Partial Differential Equations by akin to the equation over the provided boundary conditions by converting it to a scheme of algebraic equations. The algebraic equations are solved to obtain the numerical solutions to fractional partial differential equation. The finite difference schemes like Implicit, Explicit and Crank Nicholson are associated but vary in stability, exactness and performance speed. The modelling of a fractional partial differential equation

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problem, involves three major attributes: 1. Actual partial differential equation. 2. The space-time domains applicable to the fractional partial differential equation, and 3. The auxiliary boundary values and initial value conditions.

Crank Nicolson algorithm is a finite difference algorithm applied to solve diffusion differential equations. This algorithm is implicit in time, unconditionally stable and has higher order of accuracy when used for regular heat equations without any nonlinear expression coming into the equation.

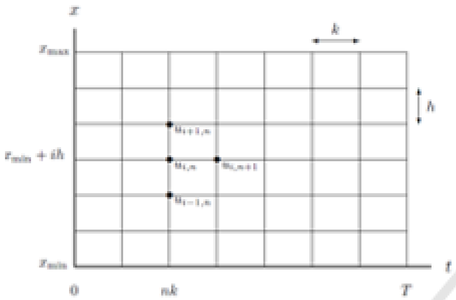


Figure 1.

# 3 RADON DIFFUSION EQUATION

The radon concentration through air medium is the outcome of the second order Radon diffusion equation which is the prime interest in this paper. Crank Nicholson finite difference algorithm has been used for solving TFRDE.

$$\frac{\partial^{\alpha}\mathrm{v}(\mathrm{x},\mathrm{t})}{\partial \mathrm{t}^{\alpha}} = \mathrm{D}\frac{\partial^{2}\mathrm{v}(\mathrm{x},\mathrm{t})}{\partial \mathrm{x}^{2}} - \lambda \mathrm{v}(\mathrm{x},\mathrm{t}),$$

where $\lambda$ is decay constant of radon.

We consider the following equation which is time fractional diffusion equation,

$$\frac{\partial^{\alpha}\mathrm{v}(\mathrm{x},\mathrm{t})}{\partial \mathrm{t}^{\alpha}} = \mathrm{D}\frac{\partial^{2}\mathrm{v}(\mathrm{x},\mathrm{t})}{\partial \mathrm{x}^{2}} - \lambda \mathrm{v}(\mathrm{x},\mathrm{t}),., \quad (1)$$

$$\text{IC: } v(\mathrm{x},0) = 0, 0 < x < l \quad (2)$$

$$\text{B.C.}: v(0,\mathrm{t}) = \mathrm{v}_0, \mathrm{t} \geq 0 \text{ and } \frac{\partial \mathrm{v}(\mathrm{x},\mathrm{t})}{\partial \mathrm{x}} = 0, \mathrm{t} \geq 0 . \quad 0 \leq \alpha \leq 1 \quad (3)$$

## 3.1 Application of Crank Nicholson Finite Difference Algorithm by Discretization

To covert the time fractional derivative in discrete form, we use $t_k = k\tau$, and $x_i = ih, \tau = \frac{T}{N}, h = \frac{l}{\mathrm{N}}$.

Let v $(x_i, t_k), i = o, 1, 2 \ldots \ldots, M \text{ and } k = o, 1, 2, \ldots \ldots, N$ be the exact solution of TFRDE from (1) - (3) at the mesh point $(x_i, t_k)$. Let $u_i^k$ be the numerical approximation of the point $\mathrm{v}(x_i, t_k) = \mathrm{v}(ih, k\tau)$.

The time fractional derivative is approximated in Caputo sense is given by,

$$\frac{\partial^{\alpha}\mathrm{v}(x_i, t_{k+1})}{\partial \mathrm{t}^{\alpha}} = \frac{1}{\Gamma(1-\alpha)}\int_0^{t_{k+1}} \frac{1}{(t_{k+1}-\xi)^{\alpha}}\frac{\partial \mathrm{v}(x_i,\xi)}{\partial \xi}\mathrm{d}\xi$$

Substitute $t_{k+1} - \xi = \eta$ and simplifying we get

$$= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{\mathrm{j}=0}^{\mathrm{k}} \frac{\left(\mathrm{v}(x_i,t_{k-j+1}) - v(x_i,t_{k-j})\right)}{\tau}\left[\mathrm{b_j}\right] + \mathrm{o}(\tau);$$

Where, $\mathrm{b_j} = (\mathrm{j}+1)^{(1-\alpha)} - (\mathrm{j})^{(1-\alpha)}, \mathrm{j} = 0,1,2 \ldots \mathrm{N}$ but $\mathrm{b}_0 = 1$, so we have;

$$\frac{\partial^{\alpha}\mathrm{v}(x_i, t_{k+1})}{\partial \mathrm{t}^{\alpha}} = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}[\mathrm{v_i^{k+1}} - \mathrm{v_i^k}[\,\mathrm{b}_0] + \mathrm{o}(\tau) + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{\mathrm{j}=1}^{\mathrm{k}}\frac{\left(\mathrm{v_i^{k-j+1}} - \mathrm{v_i^{k-j}}\right)}{\tau}[\,\mathrm{b_j}]$$

We implement central difference second order scheme in space for every interior grid point $x_i \; for \; 0 \leq i \leq m$.

$$\frac{\partial^2 \mathrm{v}(\mathrm{x},\mathrm{t})}{\partial \mathrm{x}^2} = \frac{1}{2}[\frac{v(x_{i-1},t_{k+1}) - 2v(x_i,t_{k+1}) + v(x_{i+1},t_{k+1})}{\mathrm{h}^2} + \frac{v(x_{i-1},t_k) - 2v(x_i,t_k) + v(x_{i+1},t_k)}{\mathrm{h}^2}]$$

$$\frac{\partial^2 v(\mathrm{x},\mathrm{t})}{\partial \mathrm{x}^2} = [\frac{\mathrm{v_{i-1}^{k+1}} - 2v_i^{k+1} + v_{i+1}^{k+1})}{\mathrm{h}^2} + \frac{\mathrm{v_{i-1}^{k}} - 2v_i^{k} + v_{i+1}^{k})}{\mathrm{h}^2}]$$

So the numerical approximation equation thus obtained by using the central difference and time fractional approximation the Crank Nicholson type numerical approximation to equation (1-3), expressed as follows:

$$\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[v_{\mathrm{i}}^{\mathrm{k}+1} - \mathrm{v_i^k}\right] + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{\mathrm{j}=1}^{\mathrm{k}}[\,\mathrm{b_j}]\left(v_{\mathrm{i}}^{\mathrm{k-j+1}} - v_{\mathrm{i}}^{\mathrm{k-j}}\right) = \mathrm{D}\frac{1}{2}\left[\frac{v_{\mathrm{i-1}}^{\mathrm{k+1}} - 2v_i^{k+1} + v_{i+1}^{k+1})}{\mathrm{h}^2} + \frac{\mathrm{v_{i-1}^{k}} - 2v_i^{k} + v_{i+1}^{k})}{\mathrm{h}^2}\right] - \lambda \mathrm{v}(x_i, t_k)$$

$$\left[v_i^{k+1} - v_i^k\right] + \sum_{j=1}^{k}\left[b_j\right]\left(v_i^{k-j+1} - v_i^{k-j}\right) = D\,\frac{\Gamma(2-\alpha)\tau^\alpha}{h^2}\left[\left\{v_{i-1}^{k+1} - 2v_i^{k+1} + v_{i+1}^{k+1}\right\} + \left\{v_{i-1}^{k} - 2v_i^{k} + v_{i+1}^{k}\right\}\right] - \Gamma(2-\alpha)\,\lambda v_i^k.$$

Let $r = D\,\frac{\Gamma(2-\alpha)\tau^\alpha}{h^2}$

$$and\ \mu = \Gamma(2-\alpha)\,\lambda\tau^\alpha$$

$$[v_i^{k+1} - v_i^k] + \sum_{j=1}^{k}\left[b_j\right]\left(v_i^{k-j+1} - v_i^{k-j}\right) = r\left[\left\{v_{i-1}^{k+1} - 2v_i^{k+1} + v_{i+1}^{k+1}\right\} + \left\{v_{i-1}^{k} - 2v_i^{k} + v_{i+1}^{k}\right\}\right] - \mu v_i^k \quad (4)$$

$$-rv_{i-1}^{k+1} + (1+2r)v_i^{k+1} - rv_{i+1}^{k+1} = rv_{i-1}^{k} + (1-2r-\mu)v_i^k + rv_{i+1}^k - \sum_{j=1}^{k}\left[b_j\right]\left(v_i^{k-j+1} - v_i^{k-j}\right)$$

Further, $-rv_{i-1}^{k+1} + (1+2r)v_i^{k+1} - rv_{i+1}^{k+1} = rv_{i-1}^{k} + (1-2r-\mu)v_i^k + rv_{i+1}^k - (b_1 - b_2)v_i^{k-1} - (b_2 - b_3)v_i^{k-2} - \ldots - (b_{k-1} - b_k)v_i^1 - b_k v_i^0 =$

$$= (1-\mu-b_1)\,v_i^k - rv_{i-1}^{k+1} + (1+2r)v_i^{k+1} - rv_{i+1}^{k+1} = (1-\mu-b_1)v_i^k + \sum_{j=1}^{k-1}\left(b_j - b_{j-1}\right)v_i^{k-j} + b_k v_i^0 \ \ldots.$$

where, $b_j = (j+1)^{1-\alpha} - (j)^{1-\alpha}$ ; $i = 0,1,2\ldots.m;\ k = 0,1,2,\ldots.n$

Now we convert the initial condition and boundary conditions in discritized format:

$v_i^0 = 0;\ i = 0,1,2\ldots.m;$

The boundary conditions $x_{0,}$ and $x_{m,}$ the discritization scheme implements as:

$$v_0^k = 0 \text{ and } \frac{v_{m+1}^{k+1} - v_{m-1}^{k+1}}{2h} = 0\ ;\text{implies}\ \ v_{m+1}^{k+1} = v_{m-1}^{k+1}$$

$k = 0,$

the fractional approximation IBVP looks like (From 4)

$$-rv_{i-1}^1 + (1+2r)v_i^1 - rv_{i+1}^1 = rv_{i-1}^0 + (1-2r-\mu)v_i^0 + rv_{i+1}^0 \ \ldots. \quad (6)$$

For $k \geq 0$, from 2.5

$$-rv_{i-1}^{k+1} + (1+2r)v_i^{k+1} - rv_{i+1}^{k+1} = rv_{i-1}^{k} + (1-\mu-b_1)v_i^k + rv_{i+1}^k + \sum_{j=1}^{k-1}\left(b_j - b_{j+1}\right)v_i^{k-j} + b_k v_i^0\ (k \geq 1) \ \ldots. \quad (7)$$

With initial condition $v_i^0 = 0,$

$$i = 0,1,2\ldots\ldots m \ \ldots\ldots\ldots \quad (8)$$

And boundary condition

$$v_{m+1}^{k+1} = v_{m-1}^{k+1} \quad (9)$$

The problem (6) to (9) is the complete discretized form of (1) to (3)

So, the equation can be expressed in to matrix form at $k = 0\ and\ i = 1,2,3\ldots m$

$$-rv_{i-1}^1 + (1+2r)v_i^1 - rv_{i+1}^1 = rv_{i-1}^0 + (1-2r-\mu)v_i^0 + rv_{i+1}^0$$

Can be represented in matrix form as;

$$+\begin{bmatrix} rv_0^1 \\ 0 \\ . \\ 0 \\ 0 \end{bmatrix} \ldots \quad (10)$$

$$Av^1 = Bv^0 + S$$

Now for $for\ k \geq 1;\ i = 1,2,3\ldots..m$

$$-rv_{i-1}^{k+1} + (1+2r)v_i^{k+1} - rv_{i+1}^{k+1} = rv_{i-1}^{k} + (1-\mu-b_1)v_i^k + rv_{i+1}^k + \sum_{j=1}^{k-1}\left(b_j - b_{j+1}\right)v_i^{k-j} + b_k v_i^0\ (k \geq 1)$$

$$\begin{bmatrix} (1+2r) & -r & \cdot & \cdot & & \cdot & \cdot \\ -r & (1+2r) & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & (1+2r) & & -r \\ \cdot & \cdot & \cdot & \cdot & -2r & & (1+2r) \end{bmatrix}\begin{bmatrix} v_1^1 \\ v_2^1 \\ . \\ v_{m-1}^1 \\ v_m^1 \end{bmatrix} = \begin{bmatrix} (1-2r-\mu) & r & \cdot & \cdot & & \cdot & \cdot \\ r & (1-2r-\mu) & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & (1-2r-\mu) & & r \\ \cdot & \cdot & \cdot & \cdot & 2r & & (1-2r-\mu) \end{bmatrix}\begin{bmatrix} v_1^0 \\ v_2^0 \\ . \\ v_{m-1}^0 \\ v_m^0 \end{bmatrix} + \begin{bmatrix} rv_0^1 \\ 0 \\ . \\ 0 \\ 0 \end{bmatrix} \ldots(3.10)$$

The matrix representation is given by,

$$\begin{bmatrix} (1+2r) & -r & \cdot & \cdot & & \cdot & \cdot \\ -r & (1+2r) & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & (1+2r) & & -r \\ \cdot & \cdot & \cdot & \cdot & -2r & & (1+2r) \end{bmatrix}\begin{bmatrix} v_1^{k+1} \\ v_2^{k+1} \\ . \\ v_{m-1}^{k+1} \\ v_m^{k+1} \end{bmatrix} = \begin{bmatrix} (1-2r-\mu-b_1) & r & \cdot & \cdot & & \cdot & \cdot \\ r & (1-2r-\mu-b_1) & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & (1-2r-\mu-b_1) & & r \\ \cdot & \cdot & \cdot & \cdot & 2r & & (1-2r-\mu-b_1) \end{bmatrix}$$

$$\begin{bmatrix} v_1^k \\ v_2^k \\ . \\ v_{m-1}^k \\ v_m^k \end{bmatrix} + \sum_{j=1}^{k-1}(b_j - b_{j+1}) \begin{bmatrix} v_1^{k-j} \\ v_2^{k-j} \\ . \\ v_{m-1}^{k-j} \\ v_m^{k-j} \end{bmatrix} +$$

$$(b_k) \begin{bmatrix} u_1^0 \\ u_2^0 \\ . \\ u_{m-1}^0 \\ u_m^0 \end{bmatrix} + \begin{bmatrix} ru_0^1 \\ 0 \\ . \\ 0 \\ 0 \end{bmatrix}$$

$$Av^{k+1} = Pv^{k+} + \sum_{j=1}^{k-1}(b_j - b_{j+1})v_i^{k-j} + b_k v_i^0 + S \quad \ldots.. \quad (11)$$

with nitial condition. $u_i^0 = 0$ and boundary condition $u_0^k = u_0$, $u_{m+1}^k = u_{m-1}^k$

$k = 0,1,2 \ldots..N$

represents the completely discritized matrix form of the problem.

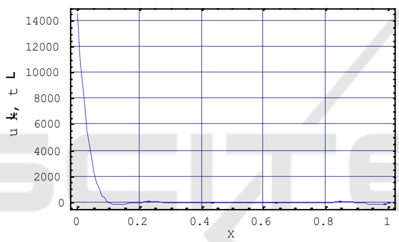


Figure 2: Radon concentration at 0.9 fractional order derivative.

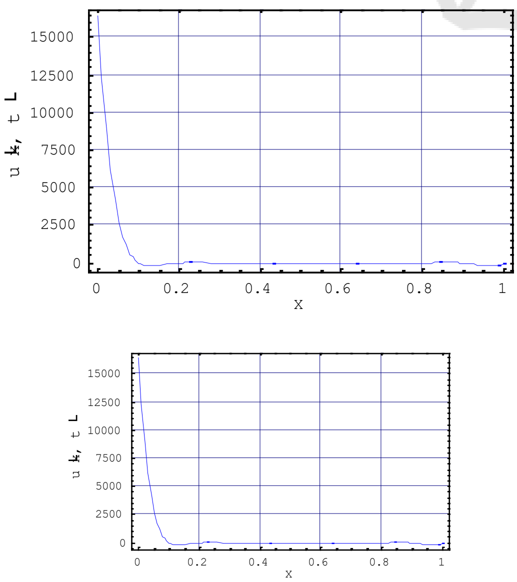


Figure 3: Radon Concentration at 0.8 fractional order derivative.

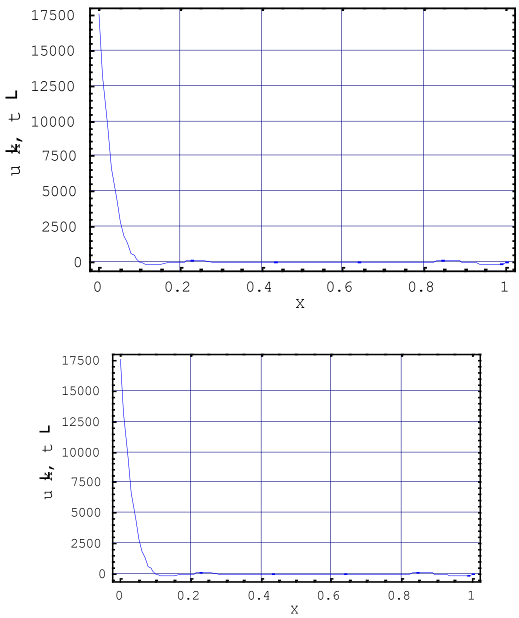


Figure 4: Radon Concentration at 0.7 fractional order derivative.

# 4 THE INVESTIGATION OF RADON DIFFUSION IN AIR BY CRANK NICHOLSON FINITE DIFFERENCE METHOD

The secondary data for different parameters included in the Radon diffusion in air medium has been referred from “Numerical and Analytical Assessment of Radon, Diffusion in Various Media and Potential of Charcoal “, as Radon Detector by (Sasaki,2006),

- The diffusion coefficient of radon in air. $D_a = 1 \times \frac{10^{-5}m^2}{s} = 0.1cm^2/s$ .
- The radon concentration in ambient air $v_0 = 200Bq/m^3$
- The radon absorption coefficient $k = \frac{1m^3}{kg}$ $and$ $\rho = \frac{1g}{cm^3}$
- The length of cylinder for measurement $l = 1m$
- The volume of cylinder for measurement $v = 1m^3$
- Radius of cylinder used for measurement is, $= \frac{1}{\sqrt{\pi}}m$ .
- The experiment for measurement of Radon diffusion was conducted for 72 hours duration, for saturation of radon activity in air.

- $v(0,t) = k\rho C_0 = 1 \times 1 \times 200 = 200 \times 10^6 Bq/cm^3$. $\lambda = 2.1 \times 10^{-6}/s$ is the decay coefficient of Radon.
- $r = \mathrm{D}\ \frac{\Gamma(2-\alpha)\tau^\alpha}{h^2}$
- $\mu = \Gamma(2-\alpha)\,\lambda$
- surface area $S = \pi r^2$

Let the fractional order derivative,
$0 \le \alpha \le 1, \alpha = 0.9, 0.8, 0.7$and
$b_j = (j+1)^{(1-\alpha)} - (j)^{(1-\alpha)}, b_0 = 1$

- For $\alpha = 0.9$, $r = 0.808004 \times 10^{-6}, S = 1m^2, \mu = 1.696 \times 10^{-8}, So, u(0,t) = 200 \times 10^6$
- For $\alpha = 0.8$, $r = 1.3254 \times 10^{-5}, S = 1m^2, \mu = 0.0278 \times 10^{-6}, So, u(0,t) = 200 \times 10^6$,
- For $\alpha = 0.7$, $r = 2.1993 \times 10^{-5}, S = 1m^2, \mu = 0.04616 \times 10^{-6}, So, u(0,t) = 200 \times 10^6$
- The estimates are used to solve the system of equations obtained in matrix form finding the relation between the radon concentration as function of distance x and time t. Similarly solving it for fractional order derivative $0 \le \alpha \le 1, \alpha = 0.9, 0.8, 0.7$we see the solution interpreted graphically by using ‘Mathematica’.

# 5 STABILITY

Lemma 5.1: If $\lambda_j(A)$;
$j = 1,2,3\ldots\ldots M-1$, represents equations of matrix A then the following results are true.
$\left|\boldsymbol{\lambda_j(A)}\right| \ge \mathbf{1}$ And $\|\mathbf{A^{-1}}\| \le \mathbf{1}$,
For $\mathbf{j = 1, 2, 3\ldots\ldots M-1}$.

Lemma 5.2: The solution obtained for the time fractional radon diffusion equation is unconditionally stable for air medium.

Proof: The stability of the solution obtained for time fractional radon diffusion equation mentioned above, we prove the relation$\|v\|_2 \le \|v^0\|_2 for\ n \ge 1.$ from (3.10) $Av^1 = Bv^0 + S$

$$v^1 = A^{-1}Bv^0 + A^{-1}S$$
$$\|v^1\|_2 = \|A^{-1}Bv\|_2 \ \le\ \|A^{-1}\|_2\|v^0\|_2\|B\|_2$$
$$\|v\|_2 \le k\|v^0\|_2\ and\ \|A^{-1}\|_2 \le 1,$$

By Principle of induction, we extend this statement for $= \mathbf{k}$ ;

$\|v\|_2 \le k\|v^0\|_2$ and for $n = k+1$ From (11)

$$Av^{k+1} = Cv^k + \sum_{j=1}^{k-1}\left(b_j - b_{j+1}\right) v_i^{k-j} + b_k v_i^0 + S$$

$$v^{k+1} = A^{-1}Cv^k + \sum_{j=1}^{k-1}\left(b_j - b_{j+1}\right)A^{-1}v_i^{k-j}$$

$$+A^{-1}b_k v_i^0 + A^{-1}S$$

$$\|v^{k+1}\|_2 \le \|C\|_2\|A^{-1}\|_2\|v^k\|_2 +$$

$$\sum_{j=1}^{k-1}\left(b_j - b_{j+1}\right)\|A^{-1}\|_2\left\|v^{k-j}\right\|_2 +$$

$$|b_k|\|A^{-1}\|_2\|v^0\|_2,$$

$$\|v^{k+1}\|_2 \le k\|v^0\|_2$$

These conditions affirm us about the unconditional stability of Crank Nicholson finite difference scheme to the Radon diffusion equation.

# 6 CONVERGENCE

The convergence of the approximate solution obtained by Crank Nicholson finite difference scheme of approximation towards the exact solution is observed here (Savovic, 2008). Let $u(x_i, t_k)$ be the exact solution of the time fractional diffusion equation in (1) to (3) and $u_i^k$ be the approximate solution for (6) to (9) at some point $(x_i, t_k)$ obtained by Crank Nicholson finite difference scheme

i =1, 2, 3.... m-1;
k =1, 2, 3……n.

$$\text{Let}e_i^k = u(x_i, t_k) - u_i^k.$$
$$E^k = (e_1^k, e_2^k \ldots \ldots\ldots e_{m-1}^k),$$
$$E^0 = 0, E_0^k = 0, E_n^k = 0.$$

$$-re_{i-1}^1 + (1+2r)e_i^1 - re_{i+1}^1 = re_{i-1}^0 + (1 - 2r - \mu)e_i^0 + re_{i+1}^0; k = 0 \ldots\ldots (6.1)$$ And

$$-re_{i-1}^{k+1} + (1+2r)e_i^{k+1} - re_{i+1}^{k+1} = re_{i-1}^k + (1 - \mu - b_1)e_i^k$$

$$+re_{i+1}^k \sum_{j=1}^{k-1}\left(b_j - b_{j+1}\right)e_i^{k-j}$$

$$+b_k e_i^0\ (k \ge 1) \ldots . (6.2)$$

$$r = \mathrm{D}\ \frac{\Gamma(2-\alpha)\tau^\alpha}{h^2}\ and\ \mu = \Gamma(2-\alpha)\,\lambda;$$
$$b_j = (j+1)^{1-\alpha} - (j)^{1-\alpha}$$

From the discretised scheme

**Lemma 6.1:** The fractional order Crank Nicholson finite difference scheme for the TFRDE $u_i^k$ converges to $u(x_i, t_k)$, the relation between the two solutions satisfies the relation $\|u(x_i, t_k) - u_i^k\| \le \|E\|_\infty + O(\tau^{1-\alpha} + h^2), i = 1,2, \ldots\ldots m-1; k = 1,2, \ldots n$.

$$\|u^{k+1}\|_2 \le |(1-\mu-b_1)|\,\|A^{-1}\|_2\|u^k\|_2 + \sum_{j=1}^{k-1}(b_j - b_{j+1})\|A^{-1}\|_2\|u^{k-j}\|_2 + |b_k|\|A^{-1}\|_2\|u^0\|_2$$

$$\|u^{k+1}\|_2 \le k\|u^0\|_2; (1-\mu) = k$$

$$\|E^1\|_\infty \le \|E^0\|_\infty + h^2[O(\tau^{1-\alpha} + O(h^2)) \ldots (1)$$

$$\|E^k\|_\infty \le \|E^0\|_\infty + h^2[O(\tau^{1-\alpha} + O(h^2)) \ldots. (2)$$

So, from (6.2) we can say.

$$|e_i^{k+1}| \le k|e_i^0| + h^2[O(\tau^{1-\alpha} + O(h^2))$$

These conditions affirm us about the unconditional convergence of Crank Nicholson finite difference scheme to the Radon diffusion equation.

# 7 DISCUSSION AND CONCLUSION

$$\|E^{k+1}\|_\infty \le \|E^0\|_\infty + h^2[O(\tau^{1-\alpha} + O(h^2)); \text{So}$$

$$\|u(x_i, t_k) - u_i^k\| \le \|E\|_\infty + O(\tau^{1-\alpha} + h^2)$$

$$|e_i^{k+1}| \le r|e_i^k| + (1 - 2r - b_1 - \mu)|e_i^k| + r|e_i^k| + (b_1 - b_2 + b_2 - b_3 + \cdots + b_{k-1} - b_k)|e_i^k| + b_k|e_i^k| + r|T_i^1|$$

$$|e_i^{k+1}| \le [r + 1 - 2r - b_1 - \mu + r + b_1 - b_k + b_k]|e_i^k| + r|T_i^1|$$

$$|e_i^{k+1}| \le [1-\mu]|e_i^k| + r|T_i^1| \;;\; |e_i^{k+1}| \le |e_i^k| + r|T_i^1|$$

$$|e_i^{k+1}| \le \|E^0\|_\infty + krh^2 o(j^{(1-\alpha)} + h^2) + r|T_i^1|$$

$$\|E^k\|_\infty \le \|E^0\|_\infty + k\tau^\alpha h^2 o(j^{(1-\alpha)} + h^2) + \tau^\alpha \Gamma(2 - \alpha) o(j^{(1-\alpha)} + h^2)$$

$$\|E^k\|_\infty \le \|E^0\|_\infty + (k+1)\tau^\alpha \Gamma(2-\alpha) o(j^{(1-\alpha)} + h^2)$$

The time fractional radon diffusion equation (1) - (3) has been solved by discretising the equation in time fractional form. The Crank Nicholson finite difference scheme has been used for approximation. The numerical solution is obtained using time fractional radon diffusion equation in air medium with initial and boundary conditions. The solution has been validated by using 'Mathematica' software. We believe the one- boundary conditions $0 < x < 1$, $0 < \alpha \le 1$, $t > 0$ initial condition: dimensional time fractional diffusion equation subjected to initial and $u_i^0 = 0$ , and boundary conditions . $u_0^k = u_0$, $u_{m+1}^k = u_{m-1}^k$. At $\alpha = 0.9, 0.8, 0.7$. The numerical solutions are analysed at $t = 0.05$ by taking into consideration the terms $\tau = 0.005$, $h = 0.1$.

Convergent numerical solution is obtained for the diffusion equation under analysis. The Radon movement and transportation through a cylinder of air is calculated as the concentration at various levels. The study of the fractional order Crank Nicholson finite difference scheme for time fractional radon diffusion equation is best fit, which gives unconditionally stable and convergent solution.

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