Efficiently Computing Maximum Clique of Sparse Graphs with Many-Core Graphical Processing Units

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Abstract: The Maximum Clique is a fundamental problem in graph theory and has numerous applications in various domains. The problem is known to be NP-hard, and even the most efficient algorithm requires significant computational resources when applied to medium or large graphs. To obtain substantial acceleration and improve scalability, we enable highly parallel computations by proposing a many-core graphical processing unit implementation targeting large and sparse real-world graphs. We developed our algorithm from CPU-based solvers, redesigned the graph preprocessing step, introduced an alternative parallelization scheme, and implemented block-level and warp-level parallelism. We show that the latter performs better when the amount of threads included in a block cannot be fully exploited. We analyze the advantages and disadvantages of the proposed strategy and its behavior on different graph topologies. Our approach, applied to sparse real-world graph instances, shows a geomean speed-up of 9x, an average speed-up of over 19x, and a peak speed-up of over 70x, compared to a parallel implementation of the BBMCSP algorithm. It also obtains a geometric mean speed-up of 1.21x and an average speed-up of over 2.0x on the same graph instances compared to the parallel implementation of the LMC algorithm.

1 INTRODUCTION

A Maximum Clique (MC) of an undirected graph $G = (V, E)$ is a subset of the vertices V such that every pair of vertices (v_i, v_j) in the subset is connected by an edge $e \in E$ and is impossible to add more vertices to the subset maintaining this property. Intuitively, an MC is the most significant fully connected subgraph within the original graph. This problem has been proved to be NP-Hard (Karp, 1972), and it is often solved with approximated strategies (Zuckerman, 1996). Nonetheless, it has multiple applications in many fields where evaluating groups of related elements is essential. Significant examples are the analysis of social network (Gschwind et al., 2015), bioinformatics (Malod-Dognin et al., 2009), coding theory (Etzion and Ostergard, 1998), and economics (Boginski et al., 2006).

Many researchers have worked to find computationally efficient algorithms. Exact solutions are often based on branch and bound procedures derived by subsequent optimizations of the Bron-Kerbosch algorithm (Bron and Kerbosch, 1973). Several approaches prune a significant section of the search space by computing an upper bound of the MC size. Many compute the bound by solving the *graph coloring* problem on the same graph. Indeed, node colors already include information on neighboring as a graph colored with n different colors cannot have a clique larger than n. Unfortunately, the coloring problem is also NP-hard. Consequently, a few approaches try to overcome this limitation by adding further analysis steps on top of the coloring procedure. For example, encoding the puzzle into a SATisfiability (SAT) (Ansótegui and Manyà, 2004) problem using an SAT-related problem called MaxSAT (Tompkins and Hoos, 2005) is particularly effective.

Graphical Processing Units (GPUs) are designed to perform parallel computations efficiently. They have numerous cores that can execute computations simultaneously, making them well-suited for tackling computationally intensive tasks (Quer et al., 2020; Cardone and Quer, 2023). Nonetheless, developing algorithms that efficiently exploit GPUs on graph problems is not trivial because of hardware constraints.

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This work focuses on an exact parallel branch and bound maximum clique algorithm that explicitly targets the manipulation of large graphs on GPUs. It is particularly efficient on sparse graphs. To the best of our knowledge, such a direction has never been thoroughly investigated in the literature in the past. We started by noticing that very few attempts have been made to bring MC algorithms to GPU; the Maximum Clique Solver Using Bitsets on GPUs (VanCompernolle et al., 2016) is somehow one of the exceptions. However, its source code is not publicly available. Therefore, we start our work from a parallel GPU implementation of the Maximal Clique Enumeration (MCE) (Almasri et al., 2023). The MCE finds all maximal cliques and can be adapted to report only the desired result. Then, we modified it by reimplementing many optimizations found in other algorithms, such as the BitBoard Maximum Clique SParse (Pablo et al., 2017) and the Large Maximum Clique (Jiang et al., 2016), applicable only to the MC problem. In particular, starting from the latter work, we redesign the vertex sorting and the graph reduction techniques. Our final preprocessing phase is more efficient and scalable, and the core algorithm uses the high parallelism of GPUs more efficiently, showing a reduced divergence among threads.

We compare our approach with three efficient algorithms for computing the Maximum Clique: The BitBoard Maximum Clique SParse (BBMCSP), the Large Maximum Clique (LMC), and the Maximum Clique Branch Reduce and Bound (MC-BRB). The results show that our approach achieves, on sparse real-world graphs, a geometric mean speed-up of 9x and an average speed-up of 19x over the parallel BBMCSP algorithm. It also obtained a geometric mean speed-up of 1.21x and an average speedup of over 2.0x compared to the parallel LMC algorithm.

Our software and all experimental results are made available on GitHub[1].

2 BACKGROUND

2.1 Graph Notation

A graph $G = (V, E)$ is a set of vertices V and a set of nodes (or links) E. Edges indicate relationships between vertices and can represent different interactions between them. Given two nodes $u, v \in V$, we identify an edge between them as $E(u, v)$. We only consider undirected graphs since the maximum clique problem is defined only for them.

The neighborhood of a vertex u, $\Gamma(u)$ is the vertices adjacent to u. Given a set of vertices $P \subseteq V$, we define $G[P]$ as the subgraph of G induced by the P vertices.

A clique is a complete graph, i.e., a graph in which each node is adjacent to all nodes. A clique is said to be *maximal* if it cannot grow by adding any other adjacent vertices. The largest maximal clique in a graph is the *maximum clique* $\omega(G)$.

A k-core of a graph G is a subgraph of G where all vertices have a degree at least equal to k. Given a graph G, we call *degeneracy* of G the size of its maximum core. The *core number* of a vertex u is the lowest k of all k-cores, including u. The *core number* of a graph G, referred to as $k(G)$, is the highest core number among all its vertices. The k-core of the graph represents an upper bound of the size of the maximum clique, i.e., $\omega(G) \leq k(G) + 1$.

Furthermore, we define the *ego network* of a vertex v, the vertex itself, and its neighborhood. The ego-network $G[N^+(v_i)]$ is the subgraph of, given an ordering, the highest-ranked neighbors of v_i.

We compute the *density* of a given undirected graph G as $d(G) = \frac{2|E|}{|V|(|V|-1)}$.

2.2 Graph Coloring

Graph coloring finds the minimum number of colors to assign to the vertices of a given graph such that every two adjacent vertices $(u, v) \in E$ have a different color, i.e., the color of u $(C(u))$ differs from the color of v $(C(v))$. Coloring can be used to compute an upper bound of the maximum clique since every node in the clique must have a different color. This property can be used to prune the search space during MC computations. The minimum number of colors needed to color a graph is called the *chromatic number* of the graph. We refer to it as $\chi(G)$. As every node in a clique must have a different color, $\chi(G)$ is an upper bound of the maximum clique, i.e., $\omega(G) \leq \chi(G)$. Since graph coloring is NP-Hard, many recent works present algorithms to find approximate solutions to the problem (Chen et al., 2023; Borione et al., 2023). Approximate methods find a coloring $\chi_{greedy}(G)$ that is an upper bound of the chromatic number $\omega(G) \leq \chi(G) \leq \chi_{greedy}(G)$.

2.3 State-of-the-Art MC Algorithms

The Bron-Kerbosh algorithm (Bron and Kerbosch, 1973) selects a vertex of the graph G and separates the remaining ones into two groups: The adjacent nodes and the non-adjacent ones. Since the maxi-

[1]https://github.com/stefanoquer/Maximum-Clique-on-GPU

mum clique is a set of fully connected vertices, the algorithm only needs to explore the adjacent nodes fully. Then, the procedure recurs on the adjoining set, which is considered a new instance of the MC problem. During the recursion, it searches for the MC between the neighboring nodes of the ones selected in the previous steps. Over the years, several optimizations have been proposed to speed up the procedure, such as reducing the vertices that need to be visited and pruning unsatisfactory branches. Furthermore, MC solvers optimized for large sparse graphs often perform a pre-processing of the input graph, order the vertices, compute an initial clique, and prune the search based on the size of the initial clique.

The BitBoard Maximum Clique SParse (Pablo et al., 2017) (BBMCSP) adopts an internal data representation based on bitsets. Bitsets encode sets to reduce the memory requirement. Moreover, they allow bit parallel sets operations, enabling efficient search and coloring.

The Large Maximum Clique (Jiang et al., 2016) (LMC) uses a k-core computation to compute an initial degeneracy-based clique. Moreover, it introduces a new pruning strategy based on incremental MaxSAT reasoning. Furthermore, it exploits the Re-NUMBER procedure (Tomita et al., 2010) to reduce the number of colors and branching vertices.

The Maximum Clique, Branch Reduce, and Bound (Chang, 2020) (MC-BRB) solves the MC problem differently. First, it adopts a greedy procedure named the Maximum Clique Degeneracy Degree (MC-DD) to compute two initial cliques, i.e., a degeneracy-based and a degree-based clique. After that, it uses a procedure called MC-EGO (which exploits information from the ego networks) to improve the maximum clique found by MC-DD and the upper bound. To compute the maximum clique, each instance of the MC problem is transformed into an equivalent task in which it is necessary to find cliques of size k on dense subgraphs. Thanks to the MC-ECO algorithm's efficiency in computing "almost" maximum cliques and the performance of the procedure to reduce the graph, the overall execution time is significantly improved.

The most noteworthy efforts to implement the MC algorithm on GPUs are the Maximum Clique Solver Using Bitsets on GPUs (VanCompernolle et al., 2016) (BBMCG) and the Parallel Maximal Clique Enumeration on GPUs (Almasri et al., 2023). BBMCG is a GPU implementation of BBMC that uses bitsets to improve its efficiency. It is suitable only for small and dense graph instances and runs out of memory for large graphs. For that reason, we do not consider it in our experimental section. The Parallel Maximal Clique Enumeration on GPUs implements a branch and bound Bron-Kerbosh approach and solves the Maximal Clique Enumeration (MCE) problem. This is a slightly different and more complex version of the MC problem. Thus, we do not consider it in our experimental section as the comparison would weigh in our favor.

3 OUR IMPLEMENTATION

3.1 Block-Wise Parallelism

To implement a fast and scalable algorithm able to manipulate large graphs even with constrained memory resources, we leverage several well-known concepts and propose a new GPU algorithm exploiting the best aspects of BBMCSP and LMC described in Section 2.3. Algorithm 1 shows our pseudo-code for computing the MC. To parallelize the process, the procedure iterates on the first level of the recursion tree (often called "the first level independent subtree") in parallel as, thanks to vertex ordering, each iteration is made independent from the following ones.

Algorithm 1: Pseudo-code for our parallel GPU-based MC algorithm.

```
   Input: Graph $G = (V, E)$
   Output: Maximum clique $C_{max}$
1  begin
2      $(G' = (V', E'), C') \leftarrow$ Pre_process (G);
3      $C_{max} \leftarrow C'$;
4      for $v_i \in |V'|$ in parallel per GPU block do
5          $P = \Gamma(v_i) \cap \{v_{i+1}, v_{i+2}, ..., v_{|V'|}\}$;
6          Compute adjacency matrix of $(G'[P])$;
7          Compute greedy coloring $\chi_{greedy}(G'[P])$;
8          $core() \leftarrow$ k-core analysis of $G'[P]$;
9          $P' = \{u \in P | core(u) + 1 \geq |C_{max}|\}$;
10         Compute adjacency matrix of $(G'[P'])$;
11         $C_{max} \leftarrow$ SearchMaxClique($G'[P']$, $P'$,
               $\{v_i\}$, $C_{max}$);
```

The algorithm initially preprocesses the input graph G (line 2), performing the following three steps. First, it exploits the parallel implementation of the MCE algorithm on GPUs to sort the vertices V of G by descending core number. Then, following the LMC algorithm (Section 2.3), it finds an initial clique representing our lower bound. Finally, it simplifies the graph G, removing all vertices with a core number smaller than the initial clique. Function Pre-process returns the reduced graph G' (not including vertices that cannot belong to the final MC) and the initial clique C' (representing our lower bound).

After that pre-processing step, we find the desired MC working on the reduced graph G'. Line 4 shows the primary iteration construct of the algorithm. This cycle takes care of the first level of the recursion tree, which is usually called a "first-level independent subtree". The iteration runs over all vertices of G' and can be easily parallelized by running an independent thread for each vertex. Unfortunately, in this approach, the P sets and the adjacency matrix must be stored for each thread. Consequently, the method requires an amount of memory proportional to the number of threads, making it unsuitable for GPUs that usually have minimal memory to store data efficiently. To solve this problem, we adopt the approach proposed by BBMCG, which is to assign each task to blocks instead of assigning tasks to threads. This strategy reduces the memory used and, simultaneously, the divergence among threads.

In line 5, the algorithm computes the intersection between the neighborhood of the selected node (v_i) of the current block and the following vertices v_i adopting the pre-computed order. Note that it is possible to precompute all the sets P before the parallel for loop and slightly optimize the process but, for clarity, we moved the computation where each task computes its own set.

In line 6, we create $G'[P]$, and in line 7, we color it using an greedy approach. This technique was initially proposed in BBMCSP. Since each GPU block consists of multiple threads, we exploit these threads to parallelize the inter-set operations typical of the coloring procedure. To speed up this process, the graph is represented as an array of bits on which accessing the neighborhood information of a node merely involves accessing the right row. The number of colors the procedure finds is used to compute the upper bound for the clique size. As a further optimization, it is possible to stop coloring when the number of colors exceeds the bound $|C_{max}| - |C|$ as no additional useful information is gained by further coloring the remaining vertices of P. The search should continue as long as the current bound can improve the best solution.

Line 8 repeats the first step performed by function Pre_process but works on the graph G' and runs only the threads in the block. After this step, it is possible to recompute a clique as in the preprocessing phase, but this procedure has been proven to be too time-consuming to achieve any benefit. Thus, in line 9, we repeat the node reduction, i.e., we remove from P any node u whose core number does not allow it to participate in a clique larger than the maximum already found.

In line 10, we create $G'[P']$, and finally, in line 11, we call the function SearchMaxClique to find the maximum clique on the subgraph including only the vertices in P'.

Algorithm 2: Our parallel function SearchMaxClique (called by Algorithm 1) to compute the final MC.

```
   Input: Graph $G = (V, E)$, Set $P$, Set $C$, Set $C_{max}$
   Output: Maximum clique $C_{max}$
1  begin
2      if $P = \emptyset$ then
3          if $|C| > |C_{max}|$ then
4              $C_{max} \leftarrow C$;
5          return $C_{max}$;
6      $B \leftarrow$ GetBranchingNodes($P$, $|C_{max}| - |C|$);
7      if $B = \emptyset$ then
8          return $C_{max}$;
9      $A \leftarrow P \setminus (B = \{b_1, b_2, ..., b_{|B|}\})$;
10     for $b_i \in B$ do
11         if first level subtree explored then
12             Donate tasks to idle blocks;
13             break;
14         $P' \leftarrow A \cup (\Gamma(b_i) \cap \{b_{i+1}, b_{i+2}, ..., b_{|B|}\})$;
15         $C_{max} \leftarrow$ SearchMaxClique($G$, $P'$,
             $C \cup \{b_i\}$, $C_{max}$);
16     return $C_{max}$;
```

Function SearchMaxClique is described by Algorithm 2.

In lines 2-5, we try to expand the current clique if the set P is empty.

Line 6 computes B, i.e., a subset of P, using the greedy coloring procedure previously presented. The nodes included in B are the ones that received a color greater than $|C_{max}| - |C|$. The presence of nodes with a color greater than this bound implies the possibility of expanding the current clique. If this set turns out to be empty, we can terminate the execution (line 7-8).

In lines 11-13, we check whether the remaining work can be shared between the idle blocks, and then we donate to each block a $(|C| + 1)$-level task. A task consists of selecting a vertex and recurring on the subproblem created by this choice. This operation dynamically rebalances the workload between threads. In line 14, we compute a new vertex set P' (including all nodes that can be used to expand the current solution) . In line 15, we recur on the MC subproblem . Since the algorithm is executed on a GPU, the recursion is performed by adopting an explicit stack recording the current state.

3.2 Warp-Wise Parallelism

As already shown in the previous section, at each recursive call of line 15, the size of the MC subproblem is reduced, and, at a certain point, there are not enough nodes to distribute the work between all threads inside a block. Thus, block-wise parallelism is effective only until the subgraph $G'' = G'[P']$ has a vertex set size $|V(G'')|$ larger or equal to 32 times the number of threads in a block. To minimize this effect, we propose the following solution. Since we can virtually guarantee enough nodes to make the block-wise parallelization worthwhile at the first recursion level, we left this level untouched, as already shown in Algorithm 1. On the contrary, Algorithm 2, from the second level subtree, receives a subproblem to be solved for each GPU warp (i.e., 32 GPU threads). Since GPU warps are the lowest hierarchical level on a GPU to possess an independent program counter, we reduce the number of nodes we can process in parallel and increase the number of instances we can solve in parallel by the same amount. This feature allows increased flexibility since we waste less computational resources when the subproblem decreases in size. Unfortunately, each instantiated subproblem requires a certain amount of memory to be allocated, increasing the memory requirements. Moreover, each warp shares the adjacency matrix of $G'[P']$. Dynamic work balancing is performed at the warp instead of the block level. It is performed internally inside the block before the first level subtree has been explored and globally among warps of different blocks after the first level subtree has been explored.

4 EXPERIMENTAL RESULTS

4.1 Experimental Setup

We ran our tests on a workstation equipped with a CPU Intel Core i9-10900KF, 64 GBytes of DDR4 RAM, and a GPU NVIDIA RTX 3070 with 8GB of onboard memory. The CPU runs with 10 physical cores and 20 threads (with hyperthreading).

All our algorithms are written in C++ and compiled with g++. The GPU implementation is written in C++/CUDA, adopting the Toolkit version 12.2.

The dataset mainly includes online graphs available at the Network Repository (Rossi and Ahmed, 2015)[2]. However, we randomly generated graphs with specific densities to compare the performances of the different algorithms in particular conditions.

Since not all parallel versions of state-of-the-art algorithms are available online, we proceed as follows to perform a significant comparison. The BBMCSP algorithm[3] has been implemented following the guidelines of the authors (Pablo et al., 2017), and we reproduced BBMCPara using the OpenMP library. LMC is available from its repository[4] in its single-threaded version. We did not modify the preprocessing strategy but parallelized the first level using the POSIX threads library. MC-BRB is only available in its single-thread version[5]. We parallelized it by following the guidelines of the original work (Chang, 2020). Both MC-EGO and MC-BRB have been parallelized using the OpenMP library. The other preprocessing procedure (like MC-DD) remains single-threaded.

4.2 Result on Random Graphs

In this section, we collect our algorithms' results on randomly generated graphs. We generated graphs of medium-large size with low-medium density to have meaningful results. This feature highlights that the pruning strategy of LMC and MC-BRB becomes more effective with higher densities. All tests have been performed using the block version of our algorithm (not the warp one). To obtain reliable data, we generated ten different graph sets sharing each value of size and density, and we presented the average run-times for each group.

Figure 1 shows the running times of each algorithm as a function of the density with graphs of different sizes. The plots show that our approach outperforms every other algorithm in practically all graph instances. In particular, our data shows that on these graph topologies, the most critical factor in reducing the execution time is the level of parallelism achievable, which significantly benefits the GPU implementation. Our data illustrates that we achieve increasing speed-ups over BBMCPara as the density decreases, going from 3x to over 10x. We also achieve a speed-up of around 2x-3x on low-density instances compared to MC-BRB. Finally, we obtained a speed-up of over 2x over LMC, often the second fastest approach for sparse graphs.

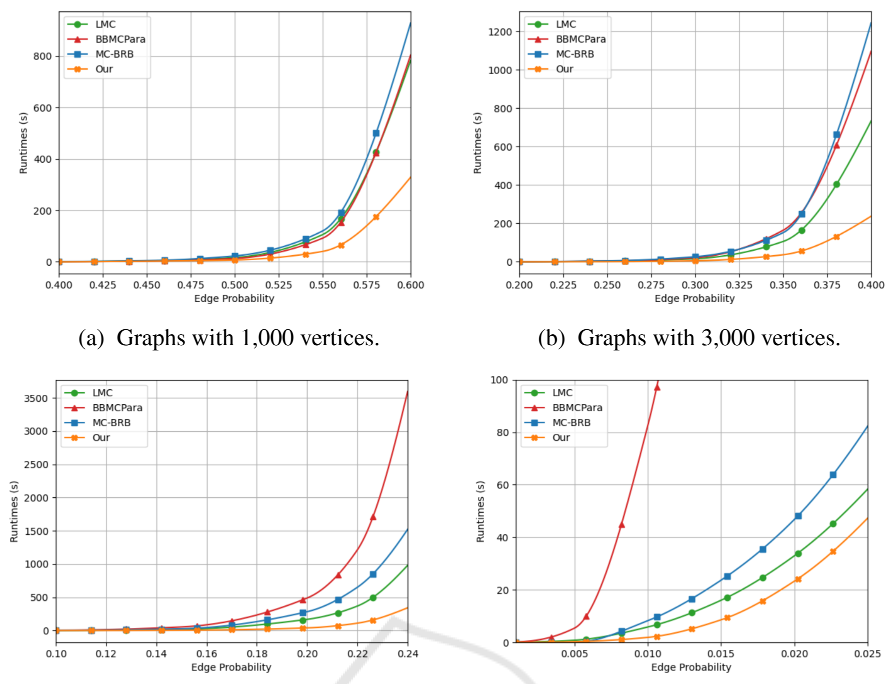
As a final remark, notice that many experiments reach the timeout while still showing a diverging behavior. This leads us to the conjecture that extending the timeout further will increase the speed-ups obtained with our tool.

[2]https://networkrepository.com/

[3]https://github.com/psanse/CliqueEnum

[4]https://home.mis.u-picardie.fr/∼cli/EnglishPage.html

[5]https://github.com/LijunChang/MC-BRB



(a) Graphs with 1,000 vertices.
(b) Graphs with 3,000 vertices.
(c) Graphs with 10,000 vertices.
(d) Graphs with 100,000 vertices.

Figure 1: Runtimes of our algorithms (on the y-axis) on random graphs as a function of the graph density (reported on the x-axis). All times are reported in seconds.

4.3 Results on Dataset Graphs

This section compares the execution time of the same algorithms analyzed in the previous section on publicly available graphs.

Table 1 collects the characteristics of our graph set. The graphs are sorted according to their average density after the reduction at the first level. We selected graphs with significantly different features to evaluate the efficiency of all algorithms with many graph topologies.

Table 2 reports all algorithms' initialization (I) and search (S) runtimes. It also reports the total (T) running times for all algorithms. Furthermore, to indicate the accuracy of the preprocessing step, it shows the value $|\omega_0|$, which represents the size of the initial clique found during the preprocessing step. *OOM* indicates an out-of-memory error, and *TO* a time-out, i.e., a running time larger than one hour. A symbol - suggests that the problem has been solved during the previous phase or has run out of memory.

As Table 2 shows, MC-BRB is the fastest algorithm on average. Even if its preprocessing time is often the highest among the approaches, its ability to find a larger initial clique and its more efficient algorithm to solve the most challenging instances produce a lower computation time. The MaxSAT approach implemented by LMC also shows excellent results in a few cases where the speed-ups are significant. Our implementation shows promising results in almost all cases when compared to BBMCSP. It also produces competitive runtimes on the lower-density graphs compared to all other approaches, making it a valid alternative with graphs locally sparse. Notice that these results are coherent with the ones presented in the previous section on random graphs. However, we can see that our algorithm pays a penalty on graphs whose density increases on lower recursion levels due to lower thread utilization and less effective pruning strategies. If we compare our solution on all graphs on which all algorithms performed a search (the first four graphs are solved during preprocessing), we are two to four times faster than the other implementations. Compared to the most dense graphs, MC-BRB manages to solve many of the graphs during the precomputation step, which results in tremendous speed-up when the other algorithms must perform a time-consuming search. Moreover, the warp-parallel version has runtimes similar to the block version in those instances where the optimization couldn't be applied; even in the worst case, the slowdown is usually minimal. The effect of the optimization is more evident in instances that take a longer execution time. In those cases, we can significantly speed up the previous version. As discussed in the last section, the warp-base

Table 1: The table shows the statistics about our dataset, i.e., number of vertices, number of edges, density, maximum and average degree of the nodes, and the size of the maximum clique.

Id	Instance	$\|V\|$	$\|E\|$	Density	Max. degree	Avg. degree	$\|\omega(G)\|$
g_{01}	co-papers-dblp	540,486	15,245,730	0.000104	3,299	56	337
g_{02}	web-uk-2002-all	18,520,486	298,113,763	2e-06	194,956	32	944
g_{03}	c-62ghs	41,731	300,538	0.000345	5,061	14	2
g_{04}	web-it-2004	509,338	7,178,414	5.5e-05	469	28	432
g_{05}	aff-orkut-user2groups	8,730,857	327,037,488	9e-06	318,268	75	6
g_{06}	rec-yahoo-songs	136,737	49,770,696	0.005324	31,431	728	19
g_{07}	soc-livejournal-user-groups	7,489,073	112,307,386	4e-06	1,053,749	30	9
g_{08}	socfb-konect	59,216,214	92,522,018	0	4,960	3	6
g_{09}	aff-digg	872,622	22,624,728	5.9e-05	75,715	52	32
g_{10}	soc-orkut	2,997,166	106,349,210	2.4e-05	27,466	71	47
g_{11}	soc-sinaweibo	58,655,849	261,321,072	0	278,491	9	44
g_{12}	wiki-talk	2,394,385	5,021,411	2e-06	100,032	4	26
g_{13}	bn-human-Jung2015_M87113878	1,772,034	76,500,873	4.9e-05	6,899	86	227
g_{14}	bn-human-BNU_1_0025864_session_2-bg	1,827,241	133,727,517	8e-05	8,444	146	271
g_{15}	soc-flickr	513,969	3,190,453	2.4e-05	4,369	12	58
g_{16}	tech-p2p	5,792,297	147,830,699	9e-06	675,080	51	178
g_{17}	bn-human-BNU_1_0025864_session_1-bg	1,827,218	143,158,340	8.6e-05	15,114	157	294
g_{18}	soc-flickr-und	1,715,255	15,555,043	1.1e-05	27,236	18	98
g_{19}	socfb-A-anon	3,097,165	23,667,395	5e-06	4,915	15	25
g_{20}	bn-human-Jung2015_M87126525	1,827,241	146,109,301	8.8e-05	8,009	160	220
g_{21}	bio-human-gene1	22,283	12,345,964	0.049731	7,940	1,108	1,335
g_{22}	bio-human-gene2	14,340	9,041,365	0.087942	7,230	1,261	1,300
g_{23}	soc-LiveJournal1	4,847,571	68,475,392	6e-06	22,887	28	321
g_{24}	web-wikipedia_link_it	2,936,413	104,673,034	2.4e-05	840,650	71	870
g_{25}	web-indochina-2004-all	7,414,866	194,109,312	7e-06	256,425	52	6,848

Table 2: Runtime statistics: The table reports all algorithms' preprocessing solution size found ($|\omega_0|$), preprocessing (I), search (S), and total time (T). The time-out (TO) is set to 3,600 seconds, i.e., one hour. The quantity of memory available is 8 GBytes of GPU RAM for our approach and 64 GBytes of system RAM for all the others. Beyond those limits, we have an out-of-memory error (OOM). A symbol - indicates that the problem has been solved during the previous phase or has run out of memory. For our algorithm, we report data for both the block-based and warp-based versions.

Id	Our						BBMCPara				LMC				MC-BRB			
	$\|\omega_0\|$	I	S (Block)	T (Block)	S (Warp)	T (Warp)	$\|\omega_0\|$	I	S	T	$\|\omega_0\|$	I	S	T	$\|\omega_0\|$	I	S	T
g_{01}	337	0.07	-	0.07	-	0.07	337	0.24	-	0.24	337	0.11	-	0.11	337	0.01	-	0.01
g_{02}	944	0.65	-	0.65	-	0.65	944	7.40	-	7.40	944	5.59	-	5.59	944	1.13	-	1.13
g_{03}	2	0.06	0.00	0.06	0.00	0.06	2	0.08	0.00	0.09	2	0.01	0.01	0.02	2	0.00	-	0.00
g_{04}	432	0.04	-	0.04	-	0.04	432	0.07	-	0.07	432	0.03	-	0.03	432	0.00	-	0.00
g_{05}	2	16.29	1.78	18.07	1.66	17.95	5	-	OOM	-	2	25.09	21.25	46.34	6	95.41	0.74	96.15
g_{06}	16	1.65	3.21	4.86	2.91	4.55	11	42.85	77.95	120.80	16	2.61	6.44	9.05	18	12.08	6.56	18.64
g_{07}	5	11.88	0.26	12.13	0.25	12.13	8	378.73	117.45	496.17	4	7.46	2.37	9.83	9	14.31	0.59	14.90
g_{08}	5	1.26	0.01	1.28	0.01	1.27	6	56.65	0.09	56.75	6	7.33	0.12	7.45	6	4.89	-	4.89
g_{09}	28	0.78	12.36	13.13	5.92	6.70	25	15.65	55.59	71.25	29	0.85	13.71	14.56	30	4.02	42.15	46.17
g_{10}	18	3.40	0.24	3.64	0.23	3.62	46	71.99	3.49	75.48	17	9.33	1.68	11.01	47	9.12	-	9.12
g_{11}	7	6.94	0.25	7.19	0.24	7.17	41	530.51	10.37	540.88	8	20.44	2.5	22.94	44	21.54	1.42	22.96
g_{12}	23	0.23	0.01	0.24	0.01	0.24	16	4.4	0.03	4.43	25	0.13	0.02	0.15	26	0.18	0.01	0.19
g_{13}	138	1.46	65.59	67.05	69.04	70.50	221	48.82	2.47	51.30	133	3.50	16.89	20.39	227	7.34	0.02	7.35
g_{14}	200	2.71	11.43	14.14	14.18	16.89	271	95.62	37.23	132.85	199	6.08	24.36	30.44	271	19.44	0.17	19.61
g_{15}	54	0.18	0.05	0.23	0.05	0.23	40	1.54	0.23	1.77	52	0.1	0.09	0.19	57	0.20	0.12	0.32
g_{16}	173	2.14	742.51	744.64	379.19	381.32	153	207.42	TO	TO	172	12.46	12.8	25.26	175	17.24	9.53	26.77
g_{17}	223	2.49	65.91	68.41	34.70	37.19	276	103.41	TO	TO	222	6.44	31.52	37.96	283	23.83	2.71	26.54
g_{18}	75	0.48	1.13	1.62	0.72	1.20	68	9.05	9.62	18.68	74	0.63	1.3	1.93	96	1.52	1.39	2.90
g_{19}	23	0.50	0.03	0.53	0.03	0.53	24	14.73	0.57	15.30	23	1.36	0.24	1.6	25	1.29	-	1.29
g_{20}	195	2.18	4.88	7.05	4.57	6.74	206	71.90	128.52	200.42	196	5.44	12.87	18.31	219	6.83	0.08	6.91
g_{21}	1,329	0.48	468.93	469.41	219.45	219.93	1,268	4.86	TO	TO	1,328	0.27	169.23	169.50	1,335	4.33	1.46	5.78
g_{22}	1,293	0.44	43.27	43.71	58.62	59.06	1,229	2.71	288.54	291.24	1,290	0.19	27.21	27.40	1,300	2.82	0.79	3.61
g_{23}	320	0.33	0.10	0.43	0.1	0.44	316	20.79	0.01	20.80	320	2.35	0.01	2.36	321	1.08	-	1.08
g_{24}	869	1.25	0.98	2.23	0.97	2.20	869	29.51	0.02	29.53	869	2.52	0.02	2.54	870	3.22	-	3.22
g_{25}	6,848	1.10	2.02	3.12	OOM	-	6,848	25.01	0.01	25.03	6,848	1.92	4.57	6.49	6,848	0.53	-	0.53

version achieves the best performances where the execution slows down because of the wasted computational resources. Thanks to this, we have speed-ups of over 2x over our block-based version.

5 CONCLUSIONS

In this paper, we focus on solving the Maximum Clique problem. Although GPU-based implementations seem an excellent path to follow to improve the scalability of the existing algorithms, very few works can be found on the topic, and, as far as we know, no implementation is available online. Consequently, starting from works on parallelizing the maximal clique enumeration on GPUs and merging contributions and ideas coming from other approaches into this methodology, we propose a GPU implementation for the Maximum Clique problem.

Our experimental analysis compares our version with the main state-of-the-art algorithms. We show that we can reach speed-ups up to 70x on graph instances that, once reduced, are dense or have a high maximum degree vs an implementation of BBMC-Para.

In future works, we still have to run our algorithm with proper implementation of the incremental MaxSAT reasoning and the incremental upped-bound approach. These contributions can further improve scalability and reduce our running times.

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