# Approximations of the Sigmoid Function Beyond the Approximation Domains for Privacy-Preserving Neural Networks

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Abstract: Artificial intelligence and data analysis have recently attracted attention, but privacy is a serious problem when sensitive data are analyezed. Privacy-preserving neural networks (PPNN) solve this problem, since they can infer without knowing any information about the input. The PPNN promotes the analyses of sensitive or confidential data and collaboration among companies by combining their data without explicitly sharing them. Fully homomorphic encryption is a promising method for PPNN. However, there is a limitation that PPNN cannot easily evaluate non-polynomial functions. Thus, polynomial approximations of activation functions are required, and much research has been conducted on this topic. The existing research focused on some fixed domain to improve their approximation accuracy. In this paper, we compared seven ways in total for several degrees of polynomials to approximate a commonly used sigmoid function in neural networks. We focused on the approximation errors beyond the domain used to approximate, which have been dismissed but may affect the accuracy of PPNN. Our results reveal the differences of each method and each degree, which help determine the suitable method for PPNN. We also found a difference in the behavior of the approximations beyond the domain depending on the parity of the degrees, the cause of which we clarified.

## 1 INTRODUCTION

In recent decades, information systems have broadly played an important role in daily life. People use smartphones, PCs and other devices, and companies provide their own systems, applications and so on. Cloud platformers help the market of information systems rapidly grow. In such situations, a significant amount of data is being generated. This leads to the era of big data, where there are many demands to perform advanced analyses on these data. Emerging technology of artificial intelligence (AI) satisfies these demands. In recent years, many AI-based data analyzing systems such as market research tools and customer relationship management systems have huge potential for improving current business and cross-domain customer analyses between companies using AI systems and opening new business avenues.

Nonetheless, people are not necessarily willing to provide their sensitive information such as health data to AI even if it will give them useful information for tasks such as disease prediction. In addition, companies and organizations tend to hesitate to share their confidential data with third-party AI systems. This reluctance also prevents collaborative analyses among several companies by combining their data, even if they provide useful insights that cannot be obtained by individual analyses.

Privacy-preserving machine learning (PPML) provides a solution to this problem of the trade-off between privacy and convenience. PPML performs training and/or inference without knowing anything about the input data. PPML can be realized using privacy-preserving computation technologies such as multiparty computation (MPC) and fully homomorphic encryption (FHE). Although both MPC and FHE enable PPML, each method has its own strength. MPC is a computation system that composes of several computing servers. Each server computes on a secret piece of data, with which the servers cannot retrieve the original data. To perform complex operations, they communicate and jointly compute. Finally, by combining the secret pieces of data, MPC can handle the data without knowing them. FHE is an encryption scheme, which enables operations on encrypted

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values without decryption. Since the data are encrypted, a server can evaluate them without knowing them and does not require communication for computation. Although MPC can compute faster than FHE, there is a risk that malicious servers may retrieve the data by jointly combining all of their secret pieces. In addition, if computing servers are owned by one company, that company can potentially retrieve the data, which forces data owners to trust the company. FHE does not have this risk because the data are encrypted by the secret key of the client. Even if a malicious server attempts to retrieve the data, they cannot decrypt the data without the key. The data owners, which are clients in this case, do not necessarily need to trust the server. Therefore, FHE-based PPML can promote joint analyses among companies since it can secrete confidential data and does not require trust in the servers.

Among many AI technologies, neural networks are fundamental and used in recent AI systems. One of major problems in implementing neural networks with FHE is how to compute activation functions. Neural networks are combination of operations on vectors and matrices and activation functions. Since activation functions are not arithmetic, FHE cannot directly adopt them because of the limitation of the operational functions. FHE allows only a few types of operation such as addition and multiplication. It requires additional processes to execute complicated operations such as divisions and conditional branches. One major method to homomorphically compute an activation function is to approximate it with polynomials. Among several types of activation functions, the sigmoid function is a popular option. It is expected to go well with polynomial approximations because it is differentiable function. Therefore, examining the accuracy of polynomial approximations of the sigmoid function is important to make neural networks with FHE more accurate.

## 1.1 Related Works

Many studies on PPML have been conducted in recent years. CryptoNets (Dowlin et al., 2016) uses homomorphic encryption called YASHE (Bos et al., 2013) to realize privacy-preserving neural networks. It employs a monomial $x^2$ as an activation function to reduce the computational complexity. The authors of (Cheon et al., 2020) proposed polynomials that approximate the sign function in the interval $[-1,1]$, with which a comparison function and ReLU function can be constructed. In 2022, the authors of (Lee et al., 2022) improved the approximating polynomial by composing several polynomials.The research of (Stoian et al., 2023) proposed neural networks that utilized a property of TFHE (Chillotti et al., 2020) called programmable bootstrapping to evaluate activation functions. This method enables evaluation of arbitrary function without decryption but is time-consuming. The proposed scheme was implemented in ConcreteML(Meyre et al., 2022). In (Trivedi et al., 2023), the authors approximated the sigmoid function with several methods and examined the errors in the approximation. They fixed the degree of approximating polynomials as three and used intervals $[-10,10]$ and $[-50,50]$ as the target ranges to approximate.

## 1.2 Our Contribution

Although previous research working on privacy-preserving neural networks has been conducted, some studies compromised the accuracy for efficiency by using a nonlinear monomial, whereas others made the approximation function only accurate inside the designated range, which we will call an approximation domain in the remainder of this paper. On the other hand, input values of activation functions may be too large or too small and lie outside the approximation domain. In that case, the error can have a non-negligible impact on the inference result of the privacy-preserving neural networks. Therefore, there is necessity to explore the behaviors of approximation functions outside the approximation domain, which can lead to more accurate privacy-preserving neural networks.

We conducted experiments on the approximations of the sigmoid function with seven types of approximations for various degrees of polynomials. We compared each method in terms of $L^2$ and $L^\infty$ errors both in the domain used for approximations and beyond the domain. Our results show that limit approximation is the best method when the parameter is set to the same value as other polynomial approximations. Our results also show that the errors outside the approximation domain behave differently depending on the approximations although those inside the domain behave similarly. By closely examining the behaviors, we clarified the cause of this difference, which provides information to avoid unexpected inference errors in privacy-preserving neural networks.

## 1.3 Organization

This paper consists of six sections including this section. In the following section, we quickly review the information used in this paper such as a fully homomorphic encryption scheme and a sigmoid function. In Section 3, four methods to approximate the sigmoid function are explained. Then, Section 4 presents the result of experiments on the accuracy of the approximations of the sigmoid function outside the range used for approximation. Section 5 discusses our experimental results. Finally, Section 6 concludes this paper.

## 2 PRELIMINARIES

In this section, we review a fully homomorphic encryption (FHE) scheme that is often used for privacy-preserving machine learning. We also review the definition of the sigmoid function, which is often used as an activation function in neural networks.

In the remainder of this paper, $\mathbb{R}$ denotes the set of real numbers.

### 2.1 Fully Homomorphic Encryption

A homomorphic encryption (HE) is an encryption scheme that enables operations on encrypted values without decryption. Technically, HE schemes consist of three functions that are identical to those of a public key encryption scheme, key generation (KeyGen), encryption (Enc), and decryption (Dec). For addition and multiplication, the following equation holds for an HE scheme: $\mathrm{Dec}(\mathrm{Enc}(m_1) \circ \mathrm{Enc}(m_2)) = m_1 \circ m_2$ for any messages $m_1$ and $m_2$ in the message space where the operation $\circ$ can be both addition and multiplication. Multiplication over two encrypted values is not straightforward. It sometimes requires a special key called or/and special operations.

While the number of homomorphic operations is limited for some HE schemes, other HE scheme do not have this limit. The former is called somewhat homomorphic encryption or leveled homomorphic encryption. The latter is called fully homomorphic encryption (FHE). A fully homomorphic encryption scheme allows an arbitrary number of operations on encrypted data using an operation called bootstrapping. Bootstrapping reduces the noise of an encrypted value, which is increased by homomorphic operations and can cause decryption failure if it exceeds a threshold. As machine learning requires a number of operations, FHE schemes are often employed to realize privacy-preserving machine learning.

Several FHE schemes have been created thus far, such as Brakerski-Gentry-Vaikuntanathan (BGV) (Brakerski et al., 2012), Cheon-Kim-Kim-Song (CKKS) (Cheon et al., 2019; Cheon et al., 2017), and a torus fully homomorphic encryption (TFHE) (Chillotti et al., 2020). All of these FHE schemes are based on the Learning With Errors (LWE) encryption scheme (Regev, 2005). The message space of BGV, BFV and TFHE is restricted to the set of integers. Strictly speaking, it is a set of integers represented by certain bits prefixed by the parameter of the scheme. Meanwhile, the message space of the CKKS scheme is the set of complex numbers. Thus, CKKS enables operations on approximate numbers, instead of integers.

TFHE has an outstanding function called programmable bootstrapping (PBS), which enables to evaluate any discrete function on encrypted data during bootstrapping without extra computation. Although this property is suitable for privacy-preserving neural networks, which require many nonlinear functions such as activation functions, TFHE can handle only integers. This does not suit for neural networks, which require decimal computations. However, CKKS can handle decimals. Although CKKS cannot execute PBS, it can rapidly evaluate polynomials. Thus, CKKS can approximate nonlinear functions through polynomial approximation.

For these reasons, both CKKS and TFHE are common options of FHE for privacy-preserving machine learning (Lou and Jiang, 2019; Meyre et al., 2022). For more information about privacy-preserving neural networks including FHE-based and multiparty computation, see (Ng and Chow, 2023). In this paper, we focus on CKKS-based privacy-preserving neural networks since they can handle decimals and approximate nonlinear functions with high accuracy.

### 2.2 Cheon-Kim-Kim-Song Scheme

As mentioned in the previous subsection, Cheon-Kim-Kim-Song (CKKS) is a fully homomorphic encryption scheme that allows operations on approximate numbers. Although CKKS has an efficient variant making use of residue number system (RNS) (Cheon et al., 2019), this modification does not essentially affect our research; we quickly explain the original CKKS (Cheon et al., 2017).

CKKS is composed of three basic operations: key generation, encryption and decryption. A sketch of these three algorithms is presented below.

- KeyGen: Sample $a, s, e$ from certain polynomial rings. Set $b := -as + e$. Output $pk := (a, b)$ as a public key and $sk := (1, s)$ as a secret key.
- $\mathrm{Enc}_{pk}(m)$: Sample $v, e_1, e_2$ from polynomial rings of small coefficients. Output $c := (vb + m + e_0, va + e_1)$ as a ciphertext for a message $m$.
- $\mathrm{Dec}_{sk}(c)$: Output $b + as$ as a decrypted message.

Since the detailed algorithms do not matter in our research, we omit them. See (Cheon et al., 2017) for details.

The addition of two encrypted values can be performed by simply adding them. However, the multiplication of two encrypted values is not straightforward. It requires an additional key called the evaluation key.

- $\mathrm{EvalKeyGen}_{sk,P}$: Sample $a', e'$ from certain polynomial rings. Output $evk := (-a's + e' + Ps^2, a)$.
- $\mathrm{Mult}_{evk,P}(\mathbf{c_1}, \mathbf{c_2})$: For $\mathbf{c_1} = (b_1, a_1)$, $\mathbf{c_2} = (b_2, a_2)$, compute $(d_0, d_1, d_2) := (b_1 b_2, a_1 b_2 + a_2 b_1, a_1 a_2)$. Output $(d_0, d_1) + \lfloor P^{-1} \cdot d_2 \cdot evk \rceil$

CKKS can efficiently compute polynomials and inverses, which enables CKKS to evaluate many functions including sigmoid function through polynomial approximation and division. Polynomials are combinations of additions and multiplications. Thus, they can be constructed via the above homomorphic addition and multiplication. To homomorphically evaluate the inverse function, approximation is required. The following equation can be used to approximate the inverse of $x$. Setting $\hat{x} := p - x$ for some $p \in \mathbb{R}$, it holds that

$$x(p+\hat{x})(p^2+\hat{x}^2)\cdots(p^{2^{r-1}}+\hat{x}^{2^{r-1}}) = p^{2^r} - \hat{x}^{2^r}. \quad (1)$$

Assuming $|x| < p/2$,

$$p^{-2^r} \cdot \prod_{k=0}^{r-1}(p^{2^k}+\hat{x}^{2^k}) = \frac{1}{x} \cdot \left(1 - \frac{\hat{x}^{2^r}}{p^{2^r}}\right) \approx \frac{1}{x}. \quad (2)$$

Thus, by computing the left-hand side of Equation (2), one obtains the approximation of $1/x$.

### 2.3 Sigmoid Function

The sigmoid function $\sigma(x)$ is a continuous function that asymptotically approaches 1 as $x$ goes to positive infinity and asymptotically approaches 0 as $x$ goes to negative infinity. This function is used as an activation function in neural networks. The sigmoid function can be expressed as follows: $\sigma(x) = \frac{1}{1+\exp(-\alpha x)}$, where $\alpha$ is a parameter. Because this parameter does not have a significant influence on the accuracy of the polynomial approximation, we set $\alpha = 1$ for the remainder of this paper, thus $\sigma(x) = 1/(1+\exp(-x))$.

To evaluate neural networks, three operations are necessary: vector addition, multiplication between a matrix and vector, and the evaluation of activation functions. Among these three operations, additions and multiplications can be performed homomorphically with the aforementioned procedures. However, the evaluation of activation functions is not straightforward since a non-arithmetic function requires additional process. Previous research on neural networks uses the approximation of activation functions. For example, (Dowlin et al., 2016) uses $x^2$ for simplicity and efficiency.

Rectified Linear Unit (ReLU) is one option of activation function, which outputs the same value as the input if it is positive and 0 if it is negative. The ReLU function cannot be expressed with Taylor expansion at the origin, which makes it require complicated procedure to approximate with polynomials. Therefore, we focus on the sigmoid function, which is infinitely differentiable.

## 3 POLYNOMIAL APPROXIMATION OF SIGMOID FUNCTION

This section explains the methods that we employed to approximate the sigmoid function. We used four approximation methods: Taylor expansion, Lagrange interpolation, Remez algorithm (Remez, 1934), and approximation of limit.

In order to approximate the sigmoid function with the above methods, there exist two major ways. One is to approximate the sigmoid function directly, and the other is to approximate via an approximation of the exponential function as we will explain in detail below.

### 3.1 Two Ways to Approximate Sigmoid Function

As mentioned above, there exist two ways to approximate the sigmoid function, directly and via exponential function. The asymptotic behaviors of the sigmoid function and polynomials differs from those of polynomials. For $x \to \infty$ (resp. $x \to -\infty$), the sigmoid function $\sigma(x) \to 1$ (resp. $\sigma(x) \to 0$). On the other hand, for $x \to \pm\infty$, $p(x) \to \pm\infty$ for any polynomial $p(x)$. Thus, it is expected that the error between the sigmoid function and approximating polynomials increases when $x$ increases or decreases.

When $x$ tends to the positive infinity, the behavior of an exponential function and polynomials is similar in that both go to the positive infinity if the leading coefficient of the polynomial is positive. Therefore, both sigmoid and approximation approach zero asymptotically when $x \to -\infty$. Thus, approximation via an exponential function is superior to the direct approximation. However, the inverse with FHE requires additional computations as explained in Section 2.2.

### 3.2 Taylor Expansion

Taylor expansion is a well-known analytic method for approximating functions. In order to execute Taylor expansion, the target function should be at least infinitely differentiable. Taylor expansion of a function $f$ at $a \in \mathbb{R}$ is expressed as $f(x) = f(a) + \frac{(x-a)}{1!}\frac{df}{dx}(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}\frac{d^{n-1}f}{dx^{n-1}}(a) + R_n$ where $R_n$ is a residue. It can be expressed as $R_n = \frac{(x-a)^n}{n!}\frac{d^n f}{dx^n}(\theta)$ where $a < \theta < x$. If $R_n$ converges as $n \to \infty$, then $f(x)$ can be written as Taylor expansion. That is, $f(x) = \sum_{n=0}^{\infty} \frac{(x-a)^n}{n!}\frac{d^n f}{dx^n}(a)$ where we set $d^0 f/dx^0 = f$ and $0! = 1$.

The sigmoid function can be written as Taylor expansion. To approximate the sigmoid function with the Taylor series, it is sufficient to use Taylor series at most degree $n$. Since the sigmoid function is symmetric at the origin, it appears accurate when it is approximated with Taylor series at the origin. In other words, the polynomial approximation $\tilde{\sigma}_n$ with Taylor series of the sigmoid function is written as $\tilde{\sigma}_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}\frac{d^k\sigma}{dx^k}(0)$. In the above approximation, the residue is cut off. In the case of the exponential function, the formula is clearer because the derivative of the exponential function is the exponential function. Therefore, the approximation $\tilde{e}_n$ of the degree $n$ is $\tilde{e}_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$.

### 3.3 Lagrange Interpolation

Lagrange interpolation is a numerical method to approximate a function with polynomials. It constructs an approximating polynomial that coincides with the target function at the given points. To construct the polynomial, one can sum up polynomials that coincide with the target function at a certain given point and take 0 at the other given points. When $n$ points $x_1, x_2, \ldots, x_n$ are given, the polynomial obtained via Lagrange interpolation of $f(x)$ is

$$p(x) := \sum_{k=1}^{n} f(x_k) \prod_{j \neq k} \frac{(x - x_j)}{(x_k - x_j)}. \tag{3}$$

When $x_1, x_2, \ldots, x_n$ are different, $p(x)$ is a polynomial of the degree of at most $n-1$, because each product part $\prod_{j \neq k}(x - x_j)/(x_k - x_j)$ is a polynomial of the degree $n-1$. In addition, $p(x_k) = f(x_k)$ holds for all $x_k \in (x_1, \ldots x_n)$. This is because the product part for $k$ is equal to 1 and for the other $j \neq k$ it is equal to 0. Therefore, in the above sense, Lagrange interpolation constructs a polynomial that approximates the target function. Note that the approximation error of Lagrange interpolation does not necessarily decrease when the degree of polynomial increases, since it is a numerical method of approximation.

### 3.4 Remez Algorithm

Remez algorithm (Remez, 1934) is also a numerical method to obtain an approximating polynomial the $L^\infty$ error of which is minimum among fixed-degree polynomials. Since $L^\infty$ norm is the maximum of a function in a certain domain, a polynomial that minimizes the error in terms of $L^\infty$ norm is called a minimax polynomial. Therefore, we can paraphrase that Remez algorithm is suitable to obtain a minimax polynomial of a specific degree.

An improved variant of this algorithm is used for RNS-CKKS bootstrapping (Lee et al., 2021). The improved variant supports the union of intervals as its approximation domain. Since the domain in which the sigmoid function is approximated is a single interval, we use the simpler one. Refer (Lee et al., 2021, Algorithm 1) for the details of the algorithm we implemented.

Remez algorithm clearly aims to minimize the error ”within” the given domain. Thus, it guarantees the accuracy inside the domain but does not guarantee the accuracy outside the approximation domain.

### 3.5 Approximation of Limit

The last approximation method used is an approximation of the limit. This method approximates an exponential function by truncating a sequence that converges to an exponential function. The sequence $\{(1 + x/n)^n\}_n$ converges to an exponential function, i.e., $\exp(x) = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$. Thus, the $n$-th term of the sequence is expected to be a good approximation for a large integer $n$.

In order to obtain the approximation of a larger number, it is efficient to repeat squaring the base. Starting from $(1 + x/2^r)$, to repeat squaring for $r$ times results $(1 + x/2^r)^{2^r}$. This is much more efficient than just multiplying the base $2^r$ times.

Note that this method is different from the other three approximations as this is not a explicit polynomial approximation. For the same parameter $n$, the other methods can obtain polynomials of the degree $n$ whereas this method implicitly yields a polynomial of degree $2^n$ if $(1 + x/2^n)^{2^n}$ is expanded. In order to compare this method to the others, one must be careful about this difference. Furthermore, this method only applies to approximations of the sigmoid function via an exponential function. This cannot be extended to direct an approximation of the sigmoid function since there does not exist a well-known

Table 1: Approximation errors inside the approximation domain $[-3,3]$.

| | $n$ | Taylor via exp | Taylor sig | Lagrange via exp | Lagrange sig | Remez via exp | Remez sig | Limit via exp | Limit sig |
|---|---|---|---|---|---|---|---|---|---|
| $L^2$ | 2 | 0.142 | 0.0899 | 4.1 | 0.0589 | 3.71 | 0.0386 | 0.0343 | - |
| | 5 | 0.0783 | 0.0336 | 0.00905 | 0.00184 | 0.00999 | 0.000996 | 0.00431 | - |
| | 8 | 0.00184 | 0.024 | 7.62e-05 | 0.000249 | 6e-05 | 0.00016 | 0.000539 | - |
| | 11 | 3.13e-05 | 0.014 | 3.2e-07 | 2.33e-05 | 1.45e-07 | 4.14e-06 | 6.74e-05 | - |
| | 14 | 2.61e-07 | 0.011 | 7.9e-10 | 3.95e-06 | 1.73e-10 | 6.65e-07 | 8.43e-06 | - |
| | 17 | 1.24e-09 | 0.00727 | 4.82e-11 | 5.65e-07 | 9e-13 | 1.72e-08 | 1.05e-06 | - |
| | 20 | 3.7e-12 | 0.00601 | 2.75e-08 | 1.25e-07 | 3.16e-14 | 2.76e-09 | 1.32e-07 | - |
| $L^\infty$ | 2 | 0.667 | 0.297 | 5.82e+03 | 0.0913 | 6.3e+03 | 0.0626 | 0.0607 | - |
| | 5 | 1.9 | 0.241 | 0.0525 | 0.00624 | 0.0429 | 0.00157 | 0.00695 | - |
| | 8 | 0.0362 | 0.22 | 0.000826 | 0.00119 | 0.000245 | 0.000252 | 0.000859 | - |
| | 11 | 0.000817 | 0.183 | 5.33e-06 | 0.000197 | 5.92e-07 | 6.51e-06 | 0.000107 | - |
| | 14 | 8.37e-06 | 0.167 | 1.79e-08 | 4.3e-05 | 7.05e-10 | 1.04e-06 | 1.34e-05 | - |
| | 17 | 4.74e-08 | 0.139 | 1.5e-09 | 8.2e-06 | 2.69e-11 | 2.7e-08 | 1.68e-06 | - |
| | 20 | 1.63e-10 | 0.126 | 7.8e-07 | 2.19e-06 | 1.08e-12 | 4.33e-09 | 2.09e-07 | - |

exponential sequence that converges to the sigmoid function as far as authors' knowledge.

## 4 EXPERIMENTS

In this section, we describe the details of our experiments and the results. In summary, the four methods explained in the previous section were implemented for both the exponential function and the sigmoid function. We conducted experiments for several degrees and approximation domains to measure the errors of approximation both inside and outside the approximation domain.

Since $\sigma(x) - 1/2$ is an odd function, polynomials of odd degrees appear appropriate for approximation. However, even polynomials were also used because the exponential function is not an odd function. To compare the direct approximations and approximations via exponential functions, it is natural to include even polynomials.

The experiments were conducted on a Mac mini with an Apple M2 CPU, 24 GB RAM and macOS Ventura 13.6.3. Python 3.10.11 was used for all experiments.

### 4.1 Details of Implementation

Since Lagrange interpolation and Remez algorithm are numerical methods, the coefficients of the approximating polynomials depend on how to implement them. We explain the details of the implementation of each method.

#### 4.1.1 Taylor Expansion

Since the Taylor expansion is an analytic method, there are few things to note. For both sigmoid and exponential functions, we used Taylor series at the origin, thus it is also known as Maclaurin expansion. The residue term $R_{n+1}$ in Taylor expansion was cut off to obtain the approximating polynomials of degree $n$. We used SymPy 1.12.1 (Meurer et al., 2017) to obtain the derivatives of both functions and substituted 0 for $x$ to obtain the coefficients.

#### 4.1.2 Lagrange Interpolation

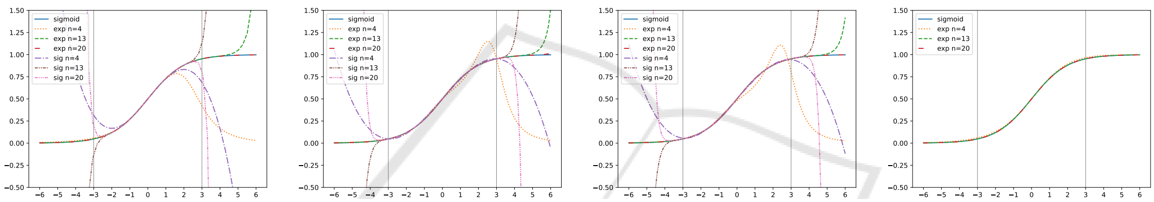
As Section 3.3 describes, Lagrange interpolation requires $n+1$ points to obtain approximating polynomials. We used the points that equally divided the approximation domain. More specifically, since we used the interval $[-3,3]$ as the approximation domain, we used $\{-3, \frac{-3n+6}{n}, \frac{-3n+2\cdot 6}{n}, \ldots, \frac{-3n+(n-1)\cdot 6}{n}, 3\}$ for $n$-th polynomial .

#### 4.1.3 Remez Algorithm

In addition to Lagrange interpolation, Remez algorithm also requires initial points to obtain an approximation function. Unlike Lagrange interpolation, the Remez algorithm requires $n+2$ points instead of $n+1$ to obtain an $n$-th polynomial. As this difference does not matter, we initialized the input points in the almost same way as Lagrange interpolation, i.e., we divided the approximation domain equally. The approximation domain was set to be identical to the Lagrange interpolation: the interval $[-3,3]$. Thus, the initial points are $\{-3, \frac{-3(n+1)+6}{n+1}, \frac{-3(n+1)+2\cdot 6}{n+1}, \ldots, \frac{-3(n+1)+n\cdot 6}{n+1}, 3\}$. The

Table 2: Approximation Errors outside the approximation domain$[-6,-3) \cup (3,6]$.

| | | Taylor | | Lagrange | | Remez | | Limit | |
|---|---|---|---|---|---|---|---|---|---|
| | $n$ | via exp | sig | via exp | sig | via exp | sig | via exp | sig |
| $L^2$ | 2 | 0.443 | 0.64 | 0.391 | 0.194 | 0.446 | 0.288 | 0.0279 | - |
| | 5 | 2.07 | 3.85 | 1.85 | 0.832 | 2.28 | 0.618 | 0.0036 | - |
| | 8 | 0.266 | 10.9 | 0.204 | 1.46 | 0.204 | 1.08 | 0.000451 | - |
| | 11 | 3.18 | 98.7 | 1.36 | 8.38 | 1.17 | 3.74 | 5.64e-05 | - |
| | 14 | 0.0145 | 3.11e+02 | 0.00687 | 16.0 | 0.00572 | 7.24 | 7.05e-06 | - |
| | 17 | 0.000641 | 3.24e+03 | 0.000243 | 1.03e+02 | 0.000241 | 28.6 | 8.81e-07 | - |
| | 20 | 1.54e-05 | 1.07e+04 | 0.00101 | 2.05e+02 | 8.02e-05 | 57.8 | 1.1e-07 | - |
| $L^\infty$ | 2 | 0.926 | 1.0 | 0.943 | 0.408 | 0.96 | 0.533 | 0.0564 | - |
| | 5 | 3.51e+03 | 12.7 | 5.65e+03 | 3.26 | 1.01e+04 | 2.48 | 0.00635 | - |
| | 8 | 0.942 | 46.3 | 0.921 | 7.34 | 0.924 | 5.53 | 0.000794 | - |
| | 11 | 2.01e+04 | 6.16e+02 | 3.51e+03 | 59.6 | 1.71e+03 | 27.7 | 9.93e-05 | - |
| | 14 | 0.206 | 2.25e+03 | 0.115 | 1.32e+02 | 0.1 | 61.9 | 1.24e-05 | - |
| | 17 | 0.0121 | 2.99e+04 | 0.00515 | 1.07e+03 | 0.0051 | 3.1e+02 | 1.55e-06 | - |
| | 20 | 0.000335 | 1.09e+05 | 0.017 | 2.36e+03 | 0.00189 | 6.94e+02 | 1.94e-07 | - |




(a) Taylor expansion (b) Lagrange interpolation (c) Remez algorithm (d) Limit approximation

Figure 1: Comparisons between the sigmoid function and approximating polynomials of the degrees 4, 13, 20.

approximation parameter $\delta$ was set to be $10^{-3}$. Additionally, the algorithm was terminated when the $L^\infty$ error fell below $10^{-10}$ to avoid the instability of the algorithm.

#### 4.1.4 Approximation of Limit

As Section 3.5 explains, this method only applies to approximations via the exponential function. Hence, we conducted experiments with this method only on approximations via the exponential function. Unlike the two previous methods, which are identical to Taylor expansion, performing this method requires only one parameter, which is the number of squares. Although this parameter is not the degree of the approximating polynomial as described in Section 3.5, we handle it similarly to the degree parameter for the other methods for comparison because these parameters are regarded as similar in the sense that they are closely related to the number of computations.

### 4.2 Accuracy Inside Approximation Domain

Here, we compare the accuracies of the obtained approximations. We use two criteria to measure the accuracy of the approximation: the $L^2$ error, which is also known as mean squared error (MSE) and $L^\infty$ error, which is the maximum absolute error in a certain range. These errors were measured inside the approximation domain $[-3,3]$. Although the Taylor expansion and limit approximation do not depend on the approximation domain, we compared all methods using identical criteria. The errors were numerically measured instead of analytically measured. We evaluated the function for $10,000$ points that uniformly separated the domain, and computed the mean squared errors for $L^2$ and took the maxima for the $L^\infty$ error respectively.

Table 1 shows the results of the approximation inside the approximation domain $[-3,3]$. In the table, ”via exp” columns show the results of approximations via the exponential function while ”sig” columns show those of direct approximation of sigmoid function. The ”$n$” column shows the degree of


(a) Approximation via exponential (b) Direct approximation

Figure 2: Comparisons among approximating polynomials of the degree 13.

the approximating polynomials. Since the limit approximation cannot be applied to the direct approximation, the column is filled with ”-”. Table 1 shows that three noteworthy points. First, the error decreases as the degree of the approximating polynomial increases for all methods of both direct and via exp. This is natural because it is expected that a polynomial of a larger degree can express broader range of functions. Second, Table 1 shows the approximation via the exponential function of the limit approximations is the best method among all approximation. Note that the comparison between the limit approximation and the other methods is not straightforward because parameter $n$ is not simply the degree of polynomials. Third, for almost all methods and degrees, the approximations via the exponential function have better $L^2$ and $L^\infty$ errors than the direct approximations of the sigmoid function.

### 4.3 Accuracy Outside Approximation Domain

To examine the errors outside the approximation domains, we used a union of intervals $[-6,-3) \cup (3,6]$ as the outside so that the lengths of the approximation domain and outside had equal lengths. When the outside is set to be broader, the errors are expected to increase because the polynomials diverge while the sigmoid function saturates.

Table 2 illustrates the $L^2$ and $L^\infty$ errors of the approximating polynomials within $[-6,-3) \cup (3,6]$ in the same manner as Table 1. These results have three noticeable points. First, the limit approximation is the best method, which is consistent with the inside case. Second, almost all results show that the approximations via the exponential function are better than the direct approximation of the sigmoid function. Third, the $L^\infty$ errors for approximations via exponential of odd degrees should be closely examined. Despite these observation, these approximations have much larger errors than the direct approximations. We discuss the details of this phenomenon in the following section.

Fig 1 shows the comparisons between the sigmoid function and the approximating polynomials of degrees 4, 13, 20. In the figure, the curves with the label ”exp $n = *$” are the polynomials of the approximation via the exponential function, those with label ”sig $n = *$” are polynomials of the direct approximations. The gray vertical lines are the bounds of the approximation domain, i.e. $-3$ and 3. Although the approximations fit the sigmoid function well within the approximation domain $[-3,3]$ for higher degrees, they do not fit well outside the domain. This figure also shows that the error grows rapidly outside the domain.

Fig 2 illustrates the comparisons among approximating polynomials of degree 13. In the figure, the gray vertical lines represent the bounds of the approximation domain, i.e., $-3$ and 3. This result shows that Taylor expansion is the worst approximation for the same degree. The Lagrange interpolation and the Remez algorithm fit fairly well, and the limit approximation has the best performance. The approximations via the exponential function in the range of negative values are quite accurate. We discuss the reasons in detail in the next section.

## 5 DISCUSSION

In this section, we discuss the details and reasons for the observed phenomena in our experiments as shown in the previous section. There are two points to discuss: the $L^\infty$ error of approximations via the exponential function of the odd degree and the difference in behaviors of approximations via the exponential function in the ranges of positive and negative values.

### 5.1 $L^\infty$ Error of Odd Degrees

As Section 4.3 shows, the $L^\infty$ error of approximations via the exponential function of the odd degree tends to be overwhelmingly larger. Here, we provide an ex-

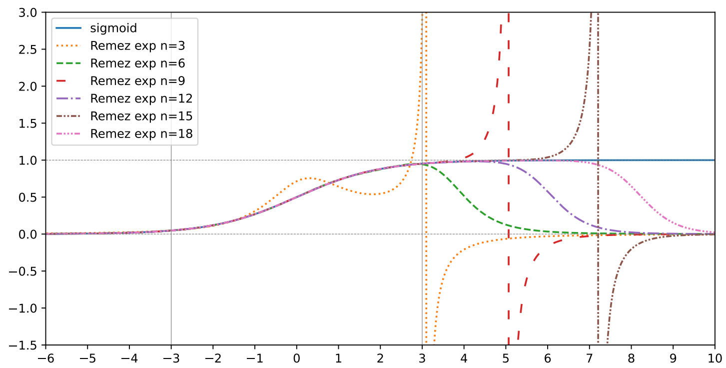



Figure 3: Polynomial approximation of odd and even degrees constructed with Remez algorithm.

planation and discuss how to overcome this issue.

Before the discussion, we must ensure that the aforementioned phenomenon is observed for other odd degrees and not observed for even degrees. Fig 3 shows the approximations via the exponential function by Remez algorithm. This figure contains the results of various degrees, half of which are odd, $3,9,15$, and the other half are even, $6,12,18$. Fig 3 illustrates that the phenomenon is observed for other odd degrees.

The reason is the division performed when the approximated sigmoid functions is constructed from the approximated exponential functions. Specifically, a polynomial $p(x)$ of an odd degree diverges to the positive infinity or negative infinity when $x \to \infty$ and to the opposite infinity when $x \to -\infty$. Thus, there exists a real number $a \in \mathbb{R}$ that satisfies $p(a) = -1$ from the continuity of the polynomial. This leads to the reason for the phenomenon for the approximated sigmoid function $\tilde{\sigma}(x)$ constructed from $p(x) \approx \exp(x)$. Since $\tilde{\sigma}(x) = 1/(1+p(-x))$, it holds $\lim_{x\nearrow -a} \tilde{\sigma}(x) = \infty$ and $\lim_{x\searrow -a} \tilde{\sigma}(x) = -\infty$. Since the exponential function rapidly increases in the positive range, the leading coefficient of approximating polynomial is set to be positive to fit this trend. Hence, the divergent point $-a$ is positive, and the approximated sigmoid functions of odd degrees diverge at certain positive values. Additionally, Fig 3 implies that the divergent point moves toward the right, i.e., to a larger value, as the degree increases. This is considered to be because the accuracy of the approximation increases even outside the approximation domain as the degree increases. Note that for even degrees, this diverging phenomenon does not occur because the approximating polynomials do not take $-1$ due to their convexity.

In Fig 3, the gray vertical lines represent the bounds of the approximation domain, $-3$ and $3$, and the gray horizontal dotted lines represent asymptotic values of the sigmoid function, 0 and 1.

## 5.2 Behavioral Differences in Positive and Negative Ranges

As mentioned in Section 4.3 and Figs 2a and 3 show, the approximations via the exponential function fit the sigmoid function well in the negative range, but not in the positive range. The reason is that the division of larger values relatively reduces the influence of the approximation error. The denominator of the sigmoid function becomes large in the negative range, e.g., $1+\exp(-(-3)) \approx 21$ and larger for smaller values. Thus, even though there is an approximation error, it will be cancelled out by the relatively large denominator. However, in the positive range the denominator is relatively small, $1+\exp(-3) \approx 1.05$ for example. Therefore, the approximation error has a significant impact on the approximated sigmoid function. This is the reason for the difference in the behaviors of the approximations via the exponential function in the positive and negative ranges.

The sigmoid functions approximated by both odd- and even-degree polynomials converge to 0 instead of 1 as $x \to \infty$. Any polynomial tends to either positive or negative infinity as $x \to -\infty$, which implies that the denominator of the approximated sigmoid function also goes to negative or positive infinity as $x \to \infty$. Therefore, the approximated sigmoid function converges to 0. This also applies to the negative range, i.e., it converges to 0 in both negative and positive ranges.

# 6 CONCLUSION

In this paper, we compared four methods in two manners to approximate the sigmoid function, which is an important component of privacy-preserving neural networks. We measured two types of errors of these methods with several degrees of polynomials and showed the relationship between accuracy and degrees of polynomials. This research also reveals the behavior of the approximated function outside the designated range to approximate, which potentially impacts on the inference result of privacy-preserving neural networks. We discuss the reasons for this unpreferable behavior. This discussion helps prevent unexpected behaviors of privacy-preserving neural networks caused by approximation errors. Overall, our results provides important knowledge about polynomial approximations of the sigmoid function that are used for FHE-based privacy-preserving neural networks.

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