Models and Algorithms for the Optimization of Multi-Period Fiber Wholesale Investments Strategies

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Keywords: Fiber Wholesale Markets, Investment Strategies, Multi-Period Portfolio Optimization, Mixed-Integer Linear Program, Valid Inequalities, Branch-and-Cut.

Abstract: This paper focuses on optimizing multi-period investment strategies for Fiber deployment. The main objective is to provide guidelines to improve the cost-effectiveness of Fiber investment strategies employed by telecommunication operators. To achieve this objective, an optimization framework is developed, providing a systematic approach to multi-period investment planning for Fiber deployment. It combines mathematical modeling and data analysis. For this, we introduce two mixed-integer linear programs to formulate the problem, taking into account demands, budget constraints and market conditions. Additionally, we propose several valid inequalities for the associated polytopes to enhance the linear relaxation and achieve tighter bounds. Relying on this modeling framework, we devise an exact optimization approach based on a Branch-and-Cut algorithm to solve the problem. Furthermore, we present a computational study that considers various instances and scenarios to assess the performance of the proposed models and algorithms.

1 INTRODUCTION

The transformation of copper access networks into Fiber optic networks is a key challenge for telecommunication operators such as Orange, in terms of economic viability, competition, and inclusion, with the aim of providing sustainable and quality telecommunication services to everyone.

1.1 Telecom Context and Motivations

In Europe, fixed broadband subscriptions are projected to increase by 25 million, rising from 260 million in 2021 to 285 million by 2026 (Dgtlinfra, 2021). Fiber is anticipated to become the dominant transmission technology, with its subscriptions growing from 30% in 2021 to over 50% by 2026 (Dgtlinfra, 2021). This transition underscores the significance of various network architecture options utilizing optical fiber, collectively referred to as Fiber To The x. Here, ”x” represents the Fiber termination point, which can be at home (FTTH), curb (FTTC), building (FTTB), antenna (FTTA), or premises (FTTP). Fiber To The x is crucial for next-generation access, significantly enhancing broadband speed and quality of service (QoS) (Dgtlinfra, 2021).

In France, the ”Plan France Très Haut Débit” (PFTHD) is designed to achieve high-speed broadband coverage across the entire national territory, targeting all homes, businesses, and public administrations by 2030. The initiative is projected to cost a total of 21 billion euros, with public investment expected to account for approximately 13 billion euros to 14 billion euros. In particular, the cost of deploying the FTTH technology across the entire territory has been estimated at several dozen billion euros by the French Senate (Angilella et al., 2016). Consequently, operators do not deploy their own FTTH networks throughout the whole territory; especially, in specific geographical areas, deployments are entrusted to third-party operators.

The key question for a commercial operator like Orange becomes defining an effective long-term strategy for aquiring optical fibers in the areas deployed by third-party operators, covering the needs of its customers while minimizing purchasing costs. The associated financial stakes amount to several hundred million euros per year, making the optimization of such strategy essential.

This study primarily focuses on optimizing the

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Hadhbi, Y., Bechler, A. and Chardy, M.
Models and Algorithms for the Optimization of Multi-Period Fiber Wholesale Investments Strategies.
DOI: 10.5220/0013182900003893

In *Proceedings of the 14th International Conference on Operations Research and Enterprise Systems (ICORES 2025)*, pages 133-145
ISBN: 978-989-758-732-0; ISSN: 2184-4372

Fiber infrastructure business decision, especially in these geographical areas where third-party operators are in charge of the Fiber deployment. This involves strategically choosing the level of investment in Fiber optic infrastructure to meet the customers'demand for high-speed and connectivity from which is increasing over time. Due to limited investment capacity, it is necessary to consider the multi-year nature of Fiber wholesale investments. This also holds significant importance because investment decisions in the telecommunication networks often have long-term implications. By developing effective investment strategies, we aim to assist telecommunication operators in maximizing the impact and benefits of their investment efforts. Moreover, the study enables telco to plan their investments strategically, considering factors such as future demand forecasts, budget constraints, so as to remain competitive in the dynamic Fiber wholesale market. This paper explores the strategic planning of Fiber optic deployment across multiple zones that are currently undeployed. Each zone has specific attributes, including the number of connectable clients and demand for connections over consecutive time periods. The operator can choose between two investment strategies: renting or co-investment of Fiber lines, with the requirement that the total deployed lines meet the demand. Zones are classified based on investment feasibility, and the decision-making process is influenced by investment percentages and cumulative investment rates, which must adhere to contractual limits. Investments incur capital and operational costs, with a budget constraint on capital expenditures. The study focuses on optimizing investment strategies to effectively meet client demands while managing costs. Moreover, this study examines several key performance indicators (KPIs) essential for evaluating the success of investment decisions in Fiber optic infrastructure. The metrics analyzed include Return on Investment (ROI), which measures the cost-effectiveness of investments by comparing generated benefits to initial costs; Capital Expenditure (CAPEX) costs, which encompass funds allocated for acquiring or upgrading infrastructure; and Operational Expenditure (OPEX) costs, reflecting ongoing expenses related to maintenance and energy consumption. Additionally, the analysis embeds Fiber renting costs, which are critical when existing Fiber lines do not meet demand, and Fiber migration costs, incurred when transitioning from rented to co-owned Fiber lines. By assessing these KPIs, the study aims to provide a comprehensive framework for informed investment decision-making in the Fiber optic sector, providing decision-makers with insights to improve the economic viability of Fiber optic deployments.

1.2 Related Works

Regarding the literature, extensive research has been conducted on the optimization of investment strategies for various energy-related challenges. These include:

- Renewable Energy Systems: studies such as those by (Wang et al., 2020),(Farah and Andresen, 2024) and (Faria et al., 2023) have explored innovative investment strategies to enhance the efficiency and sustainability of renewable energy sources.
- Power Grid Management: research by (Gao et al., 2022), (Gao et al., 2023) and (Zhang et al., 2019) has focused on optimizing investments in power grid infrastructure, aiming to improve reliability and reduce operational costs.
- Energy Efficiency: the work of (He et al., 2019) has highlighted strategies to optimize investments in energy-efficient technologies, contributing to overall energy savings.
- Smart Grids: studies by (Giannelos et al., 2023), (Tuballa and Abundo, 2016) and (Zafar et al., 2018) have examined investment optimization in smart grid technologies, emphasizing the integration of advanced communication and control systems.

Conversely, some research has investigated investments in battery storage within telecommunications networks, particularly under energy market incentives, as seen in the works of (Kerdphol et al., 2016) and (Silva et al., 2024). However, the specific area of optimizing investment strategies for Fiber deployment, particularly in FTTH networks, remains underexplored in the current state of the art. Most research on Fiber optic network design has mainly focused on network planning themselves and not the investment optimization. (Grötschel et al., 2014) studied the cost-effective deployment of optical access networks, focusing on different variants such as fiber to the home, fiber to the building, fiber to the curb, and fiber to the neighborhood. Other studies, such as (Chardy et al., 2012), (Hervet et al., 2012), (Angilella et al., 2016) and (Angilella et al., 2018), have addressed this topic, focusing on the fiber to the home. Additionally, some research has proposed optimization approaches for mobile networks, as demonstrated by (Cambier et al., 2021) and (Zappalà et al., 2022). This presents an opportunity for further investigation into investment strategies that could enhance Fiber deployment efficiency and effectiveness in telecommunications.

1.3 Contributions

In the work, we aim to provide an optimization framework for network operators to optimize their investment strategies specifically for Fiber deployment. This can be considered as the first developed framework in the literature to address this novel economic problem encountered in Fiber deployment. Our framework aims to provide an optimization approach to manage the investment strategies and minimizing costs in Fiber deployment scenarios. To achieve this, we believe that an optimization approach based on mathematical models and exact optimization algorithms can effectively solve this problem, even when dealing with large instances. For this, we provide an efficient solution to the challenges raised by the problem using mixed integer linear programming formulations (MILP), and a polyhedral approach based on a Branch-and-Cut (B&C) algorithm.

1.4 Organization of the Paper

The rest of this paper is organized as follows. In Section 2, we provide a detailed step-by-step description of the problem, which we refer to as the Multi-Period Fiber Wholesale Investment Strategies Optimization (MP-FWIS-O) problem. We consider the MP-FWIS-O as a combinatorial optimization problem and provide its intuitive formulation as a Mixed Integer Linear Program. Section 3 is dedicated to the Complexity analysis of the problem. In Section 4, we present several mathematical properties leading us to reformulate the problem, as well as a range of valid inequalities for both formulation. A Branch-and-Cut is presented in Section 5. We then present an extensive computational study in Section 6 using different classes of instances and scenarios. Finally, we summarize our results and future outlook in Section 7.

2 PROBLEM DESCRIPTION AND FORMULATION

We consider a geographical area (typically a country) composed of a set of zones Z, on which the deployment of the Fiber access network is granted to dedicated Infrastructure Operators $o_z, z \in Z$. These deployments spread over time and we thus consider a discrete time horizon $T = \{1,..,n\}$, over which we assume the future deployments to be known with certainty, denoting $D_{z,t}$ the number of Fiber lines deployed on zone $z \in Z$ up to period $t \in T \cup \{0\}$ (ie. the connectable customers), $t = 0$ denoting the last period in the past that comes before the first period in T.

We focus on a specific commercial operator of the geographical area (typically a domestic operator such as Orange) and denote by $d_{z,t}, t \in T \cup \{0\}$ the temporal evolution of its number of retail Fiber customers on zone $z \in Z$.

Considering any zone $z \in Z$ and in order to be able to satisfy its retail customers, this commercial operator has to purchase physical Fiber lines by the Infrastructure Operator of the zone o_z mixing two complementary purchasing strategies: co-investing in the deployments and renting lines. As for the co-investment strategy, we assume that the commercial operator can invest at particular periods $T^C \subset T$ when investment committees hold, and only on a specific set of investment slices denoted by I and numbered from 0 to 20 (i.e., $I = \{0,1,2,...,19,20\}$). We model the operator's co-investment decisions by binary variables $c^{upg}_{z,t,i}$, equal to 1 if the operator decides to make a new co-investment on slice $i \in I$ for zone $z \in Z$ at period $t \in T$ (0 otherwise), leading to the acquisition of a number of new Fiber lines equal to a percentage Q_i of the deployed Fiber lines for the rest of the time-horizon (note that we assume $Q_0 = 0\%$ so as to model a "no-investment" slice, indexed by 0). Specifically, the set $(Q_i)_{i \in I}$ consists of slice percentages ranging from 0% to 100% in increments of 5% (for example Q_7 is equal to 35%). At any time-period $t \in T$, based on its sequence of previous co-investments, the operator thus owns a percentage of the deployed Fiber lines $D_{z,t}$ for each zone $z \in Z$, denoted by positive continuous variable $q_{z,t} \geq 0$ (note that this variable $q_{z,t}$ will, in practice, take value within the set $(Q_i)_{i \in I}$). Introducing constant parameter $q_{z,0}$ the value of the initial cumulative investment rate at period $t = 0$, the dynamics can be described by the following equations:

$$q_{z,t} = q_{z,t-1} + \sum_{i \in I} Q_i \cdot c^{upg}_{z,t,i}, \forall z \in Z, \forall t \in T, \quad (1)$$

while ensuring that at most one slice is invested on each zone at any time-period

$$\sum_{i \in I} c^{upg}_{z,t,i} = 1, \forall z \in Z, \forall t \in T, \quad (2)$$

and that no investments are made outside committees' time-periods:

$$\sum_{i \in I \setminus \{0\}} c^{upg}_{z,t,i} = 0, \forall t \in T \setminus T^C. \quad (3)$$

Note that for regulatory motivations related to fairness (among domestic operators notably), a maximum investment rate, denoted by Q^{max}_z and assumed to be greater than $q_{z,0}$, is set on each zone $z \in Z$, such that:

$$q_{z,t} \leq Q^{max}_z, \forall t \in T. \quad (4)$$

From this dynamics, we can derive the number of co-financed Fiber lines for each zone $z \in Z$ and at each period $t \in T$, denoted by $\bar{d}_{z,t}$:

$$\bar{d}_{z,t} = D_{z,t} \cdot q_{z,t}, \forall z \in Z, \forall t \in T. \quad (5)$$

In this context, considering any zone $z \in Z$ and period $t \in T$, the whole number of retail Fiber customers of the operator must be covered either by using a part of its co-financed Fiber lines, denoted by $u^{inv}_{z,t}$, or rented lines $u^{rent}_{z,t}$:

$$u^{inv}_{z,t} + u^{rent}_{z,t} = d_{z,t}, \forall z \in Z, \forall t \in T, \quad (6)$$

knowing that the number of co-financed Fiber lines used to serve customers cannot exceed the number of co-financed Fiber lines owned by the operator:

$$u^{inv}_{z,t} \leq \bar{d}_{z,t}, \forall z \in Z, \forall t \in T. \quad (7)$$

Mixed co-investments and renting strategies induce several types of costs. First, co-investments made in zone $z \in Z$ at period $t \in T$ lead to one-shot capital expenditures (CAPEX) which are proportional to the number of Fiber lines "acquired" at this period and for this zone, denoted by positive continuous variable $capex_{z,t}$. Precisely, newly acquired Fiber lines can result from two phenomena identified in the following CAPEX cost formulation:

$$\begin{aligned} capex_{z,t} = CAPEX_{z,t} \sum_{i \in I} Q_i \cdot D_{z,t} \cdot c^{upg}_{z,t,i} \\ + Q_i \cdot \left(D_{z,t} - D_{z,t-1}\right)^{+} \cdot c^{tot}_{z,t-1,i}, \forall z \in Z, \forall t \in T, \quad (8) \end{aligned}$$

where $CAPEX_{z,t}$ stands for the unitary cost of acquiring a new Fiber line at period $t \in T$ for zone $z \in Z$, and $\left(D_{z,t} - D_{z,t-1}\right)^{+} = \max\left(0; D_{z,t} - D_{z,t-1}\right)$. For any zone $z \in Z$ we set constants $c^{tot}_{z,0,i}, \forall i \in I$ to 0 except for the unique slice $i \in I$ such that $q_{z,0} = Q_i$, which is set to 1.

On the other hand, for any zone $z \in Z$ at any period $t \in T$, the use of any co-financed Fiber line incurs an operational cost (OPEX) depending on the total co-investment slice, denoted by $SUB_{z,t,i}$ where $i \in I$ represents the index of the total co-investment slice at period t. To derive such type of cost, we first need to identify the slice associated to a given total investment rate. Introducing binary variables $c^{tot}_{z,t,i}$ equal to 1 if the cumulative investment rate is equal to $Q_i, i \in I$, we identify the unique total co-investment slice for each zone $z \in Z$ and period $t \in T$ as follows:

$$q_{z,t} = \sum_{i \in I} Q_i \cdot c^{tot}_{z,t,i}, \forall z \in Z, \forall t \in T, \quad (9)$$

$$\sum_{i \in I} c^{tot}_{z,t,i} = 1, \forall z \in Z, \forall t \in T. \quad (10)$$

Then the OPEX cost, noted $opex_{z,t}$, can be expressed as follows:

$$opex_{z,t} = \sum_{i \in I} SUB_{z,t,i} \cdot c^{tot}_{z,t,i} \cdot u^{inv}_{z,t}, \forall z \in Z, \forall t \in T. \quad (11)$$

The non-linearity of equations (11) is evident. To deal with this, we propose a new variable $f^{inv}_{z,t,i} \in \mathbf{Z}_{+}$ which denotes the number of co-owned invested and used Fiber lines when we have cumulatively the ith slice of investment in zone. Using this, we consider the inequalities (11-1)-(11-4) used to linearize and replace the quadratic equation (11). For each $z \in Z$ and $t \in T$

$$opex_{z,t} = \sum_{i \in I} SUB_{z,t,i} \cdot f^{inv}_{z,t,i}. \quad (11\text{-}1)$$

For each $z \in Z$, $i \in I$ and $t \in T$, we ensure that

$$f^{inv}_{z,t,i} \leq u^{inv}_{z,t}, \quad (11\text{-}2)$$

$$f^{inv}_{z,t,i} \leq Q_i \cdot D_{z,t} \cdot c^{tot}_{z,t,i}, \quad (11\text{-}3)$$

$$u^{inv}_{z,t} - Q_i \cdot D_{z,t} \cdot (1 - c^{tot}_{z,t,i}) \leq f^{inv}_{z,t,i}. \quad (11\text{-}4)$$

Second, the renting cost of each zone $z \in Z$ and period $t \in T$ is proportional to the number of rented lines. Let $RENT_{z,t}$ denote the unitary renting cost. The cumulative renting cost is expressed through the decision variables $rent_{z,t}$, defined as follows:

$$rent_{z,t} = RENT_{z,t} \cdot u^{rent}_{z,t}, \forall z \in Z, \forall t \in T. \quad (12)$$

Finally, at any period $t \in T$, a unitary migration cost $MIG_{z,t}$ is applied to any Fiber line rented at period $t-1$ and then served by a co-financed Fiber line at period t, in the specific case of t being the first co-investment period. Introducing the incremental demand $\Delta d_{z,t} = d_{z,t} - d_{z,t-1}$ for each zone $z \in Z$ and period $t \in T$ and denoting respectively by $u^{inv}_{z,0}$ and $u^{rec}_{z,0}$ the initial number of co-invested and rented fibers for each zone $z \in Z$ at period $t = 0$, the number of migrated fibers can be computed by distinguishing two cases:

- when the number of Fiber customers decreases from period $t-1$ to t, the number of migrated customers at t is precisely the number of customers served by co-invested fibers at t, ie. $u^{inv}_{z,t} - u^{inv}_{z,t-1}$.
- when the number of Fiber customers increases from period $t-1$ to t, the number of migrated customers at t corresponds to the decrease in the number of retail customers served with rented lines $u^{rent}_{z,t-1} - u^{rent}_{z,t}$.

This is summarized in the following equalities:

$$\begin{aligned} u^{mig}_{z,t} = \mathbb{1}_{\{\Delta d_{z,t} \leq 0\}}(u^{inv}_{z,t} - u^{inv}_{z,t-1}) \quad (13) \\ + \mathbb{1}_{\{\Delta d_{z,t} > 0\}}(u^{rent}_{z,t-1} - u^{rent}_{z,t}), \forall z \in Z, \forall t \in T. \end{aligned}$$

We notice that the first period of co-investment is, if existing, the only period $t \in T$ such that

$\sum_{i\in I\setminus\{0\}} c^{tot}_{z,t,i} - c^{tot}_{z,t-1,i} = 1$. Then the migration cost, denoted by decision variables $mig_{z,t}$ for each zone $z \in Z$ and period $t \in T$, can be bounded as follows:

$$mig_{z,t} \geq MIG_{z,t} \cdot u^{mig}_{z,t} - d_{z,t} \sum_{i\in I\setminus\{0\}} c^{tot}_{z,t,i} - c^{tot}_{z,t-1,i} \quad (14)$$

relying on the objective function to ensure its equality to 0 when needed.

Finally, we assume that the co-investments strategies are restrained by CAPEX budgets attributed to each investment committee period $t \in T^C$, denoted by $Budget_t$. Denoting P_t the set of periods between t and the next committee, we thus consider the following constraints for each committee period $t \in T^C$:

$$\sum_{z\in Z}\sum_{t'\in P_t} capex_{z,t'} \leq Budget_t. \quad (15)$$

Note that the basis of this formulation enables to consider different objective functions that are meaningful for a network operator, such as minimizing

- the total weighted sum of renting, opex and migration costs

$$\min \sum_{z\in Z}\sum_{t\in T} \alpha \cdot rent_{z,t} + \beta \cdot opex_{z,t} + \lambda \cdot mig_{z,t}, \quad (16)$$

- the total weighted sum of renting and migrated Fiber lines

$$\min \sum_{z\in Z}\sum_{t\in T} \alpha \cdot u^{rent}_{z,t} + \beta \cdot u^{mig}_{z,t}, \quad (17)$$

- the total number of not used co-financer Fiber lines

$$\min \sum_{z\in Z}\sum_{t\in T} \bar{d}_{z,t} - u^{inv}_{z,t}, \quad (18)$$

where $\alpha, \beta, \lambda \in \mathbf{R}$.

Moreover, these objective functions can be used as metrics to evaluate the solution of the problem, considering other KPIs as mentioned in the introduction.

For the sake of clarity, we consider a small-size example with one single zone (denoted by z) and a 4-periods time horizon $T = \{1,2,3,4\}$ with a unique investment committee positionned at the second period ($T^C = \{2\}$) where we choose to invest on the first slice (with $Q_1 = 5\%$). Table 1 provides a feasible solution so as to illustrate both the decision notation and the cost computation mechanisms. Note that, under the assumption that unitary renting costs are strictly greater than unitary operationnal costs ($SUB_{z,t,i} < RENT_{z,t}, \forall t \in T, \forall i \in I$), this solution is not optimal as using $u^{inv}_{z,3} = \bar{d}_{z,3} = 45$ co-financed lines at period 3 (and thus renting $u^{rent}_{z,3} = 17$ Fiber lines) instead of using $u^{inv}_{z,3} = 39$ co-financed lines (and thus renting $u^{rent}_{z,3} = 22$ Fiber lines) would lead to a lower objectif cost.

Table 1: Illustration of notation and feasible solution to the MP-FWIS-O problem.

T	0	1	2	3	4
$D_{z,t}$	0	500	800	900	1000
$d_{z,t}$	0	20	37	61	49
$c^{upg}_{z,t,i}$	-	0	$c^{upg}_{z,2,1} = 1$	0	0
$c^{tot}_{z,t,i}$	$c^{tot}_{z,0,0} = 1$	$c^{tot}_{z,2,0} = 1$	$c^{tot}_{z,2,1} = 1$	$c^{tot}_{z,3,1} = 1$	$c^{tot}_{z,4,1} = 1$
$q_{z,t}$	0%	0%	5%	5%	5%
$\bar{d}_{z,t}$	-	0	40	45	50
$capex_{z,t}$	-	0	$40\ CAPEX_{z,2}$	$5\ CAPEX_{z,3}$	$5\ CAPEX_{z,4}$
$u^{inv}_{z,t}$	0	0	37	39	49
$opex_{z,t}$	-	0	$37\ SUB_{z,2,1}$	$39\ SUB_{z,3,1}$	$49\ SUB_{z,4,1}$
$u^{rent}_{z,t}$	0	20	0	22	0
$rent_{z,t}$	-	$20\ RENT_{z,1}$	0	$22\ RENT_{z,3}$	0
$u^{mig}_{z,t}$	-	0	20	0	10
$mig_{z,t}$	-	0	$20\ MIG_{z,2}$	0	0

In this study, our focus is primarily on the objective function (16) with α, β and λ equal to 1, as these 3 types of costs are operationnal costs.

In additon, we will denote $\mathcal{P}(Z,T,T^C,I,Q,D,d,Budget)$ the polytope associated to (in)equations (1) to (15).

In the following section, we will study the complexity of the MP-FWIS-O problem.

3 PROBLEM COMPLEXITY

From a complexity point of view, the MP-FWIS-O problem becomes polynomial when the budget constraints are relaxed and the objective is to minimize certain costs such as the total renting cost, the total migration cost and the total OPEX cost. In such scenarios, the problem can be efficiently and optimally solved in polynomial time (Cook, 1971). However, when facing strict budget constraints, the problem becomes more complex, and finding an optimal solution within polynomial time becomes challenging. In general, we believe that this problem is NP-hard, meaning that it is computationally difficult to solve optimally in polynomial time (Cook, 1971). In particular, the knapsack problem can be viewed as a specific case of the problem when limited to a single investment committee period. The knapsack problem is a well-known combinatorial optimization problem, where the goal is to select a subset of items with maximum value while respecting a capacity constraint. The knapsack problem has shown to be NP-hard. It can be seen as a mathematical representation of the budget-constraint. The knapsack's capacity corresponds to the budget constraint, and the items represent various investment options or slices that can be chosen within that constraint. The objective is to find the combination of items that maximizes the overall value or utility, similar to how one would aim to optimize multiple costs within a limited budget.

4 KEY ENHANCEMENTS

In this section, we investigate properties of the MP-FWIS-O problem and derive potential enhancements for the basic formulation provided in Section 2, leading us to propose a reformulation for the problem. In addition we provide valid inequalities for both formulations.

4.1 Properties on Optimal Solutions

First, we introduce some properties related to the optimality of solutions.

Proposition 1. *For each zone $z \in Z$, period $t \in T$ and slice $i \in I$, let us define*

$$D_{z,t,i} = \frac{\lfloor Q_i \cdot D_{z,t} \rfloor}{Q_i}.$$

Then, equation

$$\bar{d}_{z,t} = \sum_{i \in I} Q_i \cdot D_{z,t,i} \cdot c^{tot}_{z,t,i}, \forall z \in Z, \forall t \in T \qquad (5.1)$$

is valid for the polytope $\mathcal{P}(Z,T,T^C,I,Q,D,d,Budget)$.

Note that integrating equations (5.1) instead of (5) in the formulation ensures the integrality of variables $\bar{d}$.

Proposition 2. *Assuming a hierarchy in costs parameters (typically we have $SUB_{z,t,i} < RENT_{z,t}$ $\forall z \in Z$ $\forall t \in T$, $\forall i \in I$), the integrality of variables u^{inv}, u^{rent} and u^{mig} is ensured in any optimal solution when c^{upg} and c^{tot} are integers.*

Based on this property, we will replace constraints (5) by (5.1) and relax the integrality constraints on the variables $\bar{d}$, u^{inv}, u^{rent} and u^{mig} in the reformulation given in Section 4.3.

Proposition 3. *For any zone $z \in Z$, let us define*

$$\tilde{Q}_z = \min\{Q_i, i \in I : \max_{t \in T} \frac{d_{z,t}}{D_{z,t}} \leq Q_i\}.$$

Then, assuming strictly positive unitary CAPEX costs ($CAPEX_{z,t} > 0, \forall z \in Z, \forall t \in T$), there exists an optimal solution which satisfies:

$$q_{z,t} \leq \max(\tilde{Q}_z; q_{z,0}),\ \forall z \in Z, \forall t \in T.$$

In the rest of the article, we define $\bar{q}_z = \min(Q^{max}_z; \max(\tilde{Q}_z; q_{z,0}))$.

4.2 Compactness

For the reformulation proposed in Section 4.3, we aim at decreasing the number of variables and constraints compared to the one provided in Section 2. For this, we consider variables $c^{upg}_{z,t,i}$, $c^{tot}_{z,t,i}$, $q_{z,t}$, $u^{mig}_{z,t}$ and $mig_{z,t}$ only for periods in T^C. Therefore, we introduce function the following fonction C which provides the period corresponding to latest committee anterior to t within the time horizon (and 0 if no commitee occured):

$$C : \left| \begin{array}{ccl} T & \longrightarrow & T^C \cup \{0\} \\ t & \longmapsto & \begin{cases} 0 \text{ if } T^C \cap \{1,...,t\} = \emptyset \\ \max[T^C \cap \{1,...,t\}] \text{ otherwise} \end{cases} \end{array} \right.$$

On the other hand, several former constraints need consequently to be modified accordingly. The following propositions present reformulated constraints that will be utilized in the reformulation, while remaining valid for $\mathcal{P}(Z,T,T^C,I,Q,D,d,Budget)$.

Proposition 4. *Consider a zone $z \in Z$ and a period $t \in T^C$. Then, the following inequalities*

$$q_{z,t} = q_{z,C(t-1)} + \sum_{i \in I} Q_i \cdot c^{upg}_{z,t,i} \qquad (\tilde{1})$$

$$q_{z,t} \leq \bar{q}_z \qquad (\tilde{4})$$

$$\begin{aligned} capex_{z,t} = CAPEX_{z,t} \sum_{i \in I} Q_i \cdot D_{z,t} \cdot c^{upg}_{z,t,i} \\ +Q_i \cdot \left(D_{z,t} - D_{z,t-1}\right)^+ \cdot c^{tot}_{z,C(t-1),i} \end{aligned} \qquad (8.1)$$

$$\begin{aligned} u^{mig}_{z,t} = \mathbb{1}_{\{\Delta d_{z,t} \leq 0\}} (u^{inv}_{z,t} - u^{inv}_{z,t-1}) \\ + \mathbb{1}_{\{\Delta d_{z,t} > 0\}} (u^{rent}_{z,t-1} - u^{rent}_{z,t}) \end{aligned} \qquad (\tilde{13})$$

$$mig_{z,t} \geq MIG_{z,t} \cdot u^{mig}_{z,t} - d_{z,t} \sum_{i \in I \setminus \{0\}} c^{tot}_{z,t,i} - c^{tot}_{z,C(t-1),i} \qquad (\tilde{14})$$

are valid for the polytope $\mathcal{P}(Z,T,T^C,I,Q,D,d,Budget)$.

Proposition 5. *Consider a zone $z \in Z$ and a period $t \in T \setminus T^C$. Then, the following inequalities*

$$\bar{d}_{z,t} = \sum_{i \in I} Q_i \cdot D_{z,t,i} \cdot c^{tot}_{z,C(t),i}, \qquad (\tilde{5})$$

$$capex_{z,t} = CAPEX_{z,t} \sum_{i \in I} Q_i \cdot \left(D_{z,t} - D_{z,t-1}\right)^+ \cdot c^{tot}_{z,C(t-1),i}, \qquad (8.2)$$

$$opex_{z,t} = \sum_{i \in I} SUB_{z,t,i} \cdot c^{tot}_{z,C(t),i} \cdot u^{inv}_{z,t}, \qquad (\tilde{11})$$

are valid for the polytope $\mathcal{P}(Z,T,T^C,I,Q,D,d,Budget)$.

Remark1: Inequalities (4) have been revised at the light of Proposition 3, incorporating the changes made by considering only periods T^C.
Remark2: Equation (8) needs to be modified by considering it only for $t \in T^C$ and introducing additional equations for the rest of periods in $T \setminus T^C$ as follows:

$$capex_{z,t} = CAPEX_{z,t} \sum_{i \in I} Q_i \cdot \left(D_{z,t} - D_{z,t-1}\right)^+ \cdot c^{tot}_{z,t-1,i}. \tag{8.2}$$

4.3 Reformulation

Based on the previous results, we reformulate the MP-FWIS-O problem as follows:

$$\min \sum_{z \in Z} \sum_{t \in T} rent_{z,t} + opex_{z,t} + \sum_{t \in T^C} mig_{z,t}, \tag{$\tilde{16}$}$$

subject to

$$q_{z,t} = q_{z,C(t-1)} + \sum_{i \in I} Q_i \cdot c^{upg}_{z,t,i}, \forall z \in Z, \forall t \in T^C, \tag{$\tilde{1}$}$$

$$\sum_{i \in I} c^{upg}_{z,t,i} = 1, \forall z \in Z, \forall t \in T^C, \tag{$\tilde{2}$}$$

$$q_{z,t} \leq \bar{q}_z, \forall z \in Z, \forall t \in T^C, \tag{$\tilde{4}$}$$

$$\bar{d}_{z,t} = \sum_{i \in I} Q_i \cdot D_{z,t,i} \cdot c^{tot}_{z,C(t),i}, \ \forall z \in Z \ \forall t \in T, \tag{$\tilde{5}$}$$

$$u^{inv}_{z,t} + u^{rent}_{z,t} = d_{z,t}, \forall z \in Z, \forall t \in T, \tag{6}$$

$$u^{inv}_{z,t} \leq \bar{d}_{z,t}, \forall z \in Z, \forall t \in T, \tag{7}$$

$$capex_{z,t} = CAPEX_{z,t} \sum_{i \in I} Q_i \cdot D_{z,t} \cdot c^{upg}_{z,t,i}$$
$$+ Q_i \cdot \left(D_{z,t} - D_{z,t-1}\right)^+ \cdot c^{tot}_{z,C(t-1),i}, \forall z \in Z, \forall t \in T^C, \tag{8.1}$$

$$capex_{z,t} = CAPEX_{z,t} \sum_{i \in I} Q_i \cdot \left(D_{z,t} - D_{z,t-1}\right)^+ \cdot c^{tot}_{z,C(t-1),i},$$
$$\forall z \in Z, \forall t \in T \setminus T^C, \tag{8.2}$$

$$q_{z,t} = \sum_{i \in I} Q_i \cdot c^{tot}_{z,t,i}, \forall z \in Z, \forall t \in T^C, \tag{$\tilde{9}$}$$

$$\sum_{i \in I} c^{tot}_{z,t,i} = 1, \forall z \in Z, \forall t \in T^C, \tag{$\tilde{10}$}$$

$$opex_{z,t} = \sum_{i \in I} SUB_{z,t,i} \cdot c^{tot}_{z,C(t),i} \cdot u^{inv}_{z,t}, \forall z \in Z, \forall t \in T, \tag{$\tilde{11}$}$$

$$rent_{z,t} = RENT_{z,t} \cdot u^{rent}_{z,t}, \forall z \in Z, \forall t \in T, \tag{12}$$

$$u^{mig}_{z,t} = \mathbb{1}_{\{\Delta d_{z,t} \leq 0\}} (u^{inv}_{z,t} - u^{inv}_{z,t-1}) + \mathbb{1}_{\{\Delta d_{z,t} > 0\}} (u^{rent}_{z,t-1} - u^{rent}_{z,t}),$$
$$\forall z \in Z, \forall t \in T^C, \tag{$\tilde{13}$}$$

$$mig_{z,t} \geq MIG_{z,t} \cdot u^{mig}_{z,t} - d_{z,t} \sum_{i \in I \setminus \{0\}} c^{tot}_{z,t,i} - c^{tot}_{z,C(t-1),i},$$
$$\forall z \in Z, \forall t \in T^C, \tag{$\tilde{14}$}$$

$$\sum_{z \in Z} \sum_{t' \in P_t} capex_{z,t'} \leq Budget_t, \forall z \in Z, \forall t \in T^C, \tag{15}$$

$$c^{upg}_{z,t,i}, c^{tot}_{z,t,i} \in \{0,1\}, \forall z \in Z, \forall t \in T^C, \forall i \in I,$$

$$capex_{z,t}, rent_{z,t}, opex_{z,t}, u^{rent}_{z,t}, u^{inv}_{z,t} \geq 0, \forall z \in Z, \forall t \in T,$$

$$q_{z,t}, u^{mig}_{z,t}, mig_{z,t} \geq 0, \forall z \in Z, \forall t \in T^C.$$

Note that equations ($\tilde{11}$) need to be linearized, as previously mentioned.

The formulations presented in Section 2 and in this section will be respectively referred to as MILPI and MIPLII. Moreover, let $\tilde{\mathcal{P}}(Z,T,T^C,I,Q,D,d,Budget)$ denote the polytope associated with Formulation MILPII, representing the convex hull of solutions obtained by satisfying all the constraints previously presented in our reformulation, while imposing integrality constraints for certain variables.

4.4 Valid Inequalities

In what follows, we will introduce several valid inequalities to enhance the linear relaxation of our formulations. These inequalities are intended to strengthen the bounds of the linear relaxation. By incorporating these additional constraints, we can achieve more accurate and efficient solutions.

Based on inequalities (4), we introduce the following inequality ensuring the non selection of certain slices i in I having a Q_i bigger than the maximum investement rate $\bar{q}_z$.

Proposition 6. *Consider a zone $z \in Z$. Then, the following inequality*

$$\sum_{t \in T^C} \sum_{\substack{i \in I \\ Q_i > \bar{q}_z}} c^{tot}_{z,t,i} + \sum_{\substack{i \in I \\ Q_i + q_{z,0} > \bar{q}_z}} c^{upg}_{z,t,i} = 0, \tag{19}$$

is valid for the polytope $\mathcal{P}(Z,T,T^C,I,Q,D,d,Budget)$.

Proposition 7. *Consider a zone $z \in Z$ and a period $t \in T$. Let i be a slice in I. Then, the following inequalities*

$$c^{tot}_{z,t,i} \leq \sum_{\substack{i' \in I \\ Q_{i'} \geq Q_i}} c^{tot}_{z,t+1,i'}, \tag{20}$$

$$c^{tot}_{z,t,i} + \sum_{\substack{i' \in I \\ Q_{i'} > Q_i}} c^{tot}_{z,t-1,i'} \leq 1, \tag{21}$$

$$c^{tot}_{z,t,i} + \sum_{\substack{i' \in I \\ Q_{i'} < Q_i}} c^{tot}_{z,t+1,i'} \leq 1, \tag{22}$$

are valid for the polytope $\mathcal{P}(Z,T,T^C,I,Q,D,d,Budget)$.

They ensure the growth in the co-investment rate between each two consecutive periods. For this, we propose further valid inequalities used to ensure the growth in the co-investment rate during the entire temporal horizon T.

Proposition 8. *Consider a zone $z \in Z$ and two periods $t, t' \in T$ with $t' > t$. Let i be a slice in I. Then, the* *following inequality*

$$c^{tot}_{z,t,i} + \sum_{\substack{i' \in I \\ Q_{i'} < Q_i}} c^{tot}_{z,t',i'} \leq 1, \qquad (23)$$

is valid for the polytope $\mathcal{P}(Z,T,T^C,I,Q,D,d,Budget)$.

Based on this, we introduce a *conflict graph* $G^z_{I,T}$ for each zone $z \in Z$. Two nodes $v_{i,t}$ and $v_{i',t'}$ are linked by an edge in $G^z_{I,T}$ if $t = t'$ or $t' > t$ and $Q_i > Q_{i'}$. Using this, we introduce the following *clique-based* inequalities.

Proposition 9. *Consider a zone* $z \in Z$*. Let* C *be a clique in the conflict graph* $G^z_{I,T}$*. Then, the following inequality*

$$\sum_{v_{i,t} \in C} c^{tot}_{z,t,i} \leq 1, \qquad (24)$$

is valid for the polytope $\mathcal{P}(Z,T,T^C,I,Q,D,d,Budget)$.

Using the same conflict graph, we introduce the so-called *odd-cycle* inequalities.

Proposition 10. *Consider a zone* $z \in Z$*. Let* H *be an odd-cycle in the conflict graph* $G^z_{I,T}$*. Then, the following inequality*

$$\sum_{v_{i,t} \in H} c^{tot}_{z,t,i} \leq \frac{|H|-1}{2}, \qquad (25)$$

is valid for the polytope $\mathcal{P}(Z,T,T^C,I,Q,D,d,Budget)$.

We propose also another conflict graph for each zone $z \in Z$ which is related to the rate max $\bar{q}_z$ constraint (inequalities (4)). For this, we consider a *conflict graph* G^{rate}_z. Two node $v_{i,t}$ and $v_{i',t'}$ are linked by an edge in G^{rate}_z if $Q_i + Q_{i'} + q_{z,0} > \bar{q}_z$.

Proposition 11. *Consider a zone* $z \in Z$*. Let* $v_{i,t}$ *and* $v_{i',t'}$ *be two linked nodes in the conflict graph* G^{rate}_z*. Then, the following inequality*

$$c^{upg}_{z,t,i} + c^{upg}_{z,t',i'} \leq 1, \qquad (26)$$

is valid for the polytope $\mathcal{P}(Z,T,T^C,I,Q,D,d,Budget)$.

Using this, we introduce some clique inequalities as already done for the conflict graph $G^z_{I,T}$.

Proposition 12. *Consider a zone* $z \in Z$*. Let* C *be a clique in the conflict graph* G^{rate}_z*. Then, the following inequality*

$$\sum_{v_{i,t} \in C} c^{upg}_{z,t,i} \leq 1, \qquad (27)$$

is valid for the polytope $\mathcal{P}(Z,T,T^C,I,Q,D,d,Budget)$.

Proposition 13. *Consider a zone* $z \in Z$*. Let* H *be an odd-cycle in the conflict graph* G^{rate}_z*. Then, the following inequality*

$$\sum_{v_{i,t} \in H} c^{upg}_{z,t,i} \leq \frac{|H|-1}{2}, \qquad (28)$$

is valid for the polytope $\mathcal{P}(Z,T,T^C,I,Q,D,d,Budget)$.

On the other hand, we propose some valid inequalities to ensure the non selection of certain investment slices having a rate smaller or greater than the current investment rate at each period and for each zone.

Proposition 14. *Consider a zone* $z \in Z$ *and a period* $t \in T$*. Let* i *be a slice in* I*. Then, the following inequality*

$$\sum_{\substack{i' \in I \\ Q_{i'} < Q_i}} c^{tot}_{z,t,i'} + c^{upg}_{z,t,i} \leq 1, \qquad (29)$$

is valid for the polytope $\mathcal{P}(Z,T,T^C,I,Q,D,d,Budget)$.

This strengthen the link between the investment upgrade slice and the cumulative investment slice at each period.

Proposition 15. *Consider a zone* $z \in Z$ *and a period* $t \in T$*. Then, the following inequality*

$$\sum_{i \in I} f^{inv}_{z,t,i} = u^{inv}_{z,t}, \qquad (30)$$

is valid for the polytope $\mathcal{P}(Z,T,T^C,I,Q,D,d,Budget)$.

This ensures that the number of co-owned invested Fiber lines used in zone $z \in Z$ at period $t \in T$ is equal to the total number of co-owned invested Fiber lines used over all slices $i \in I$ in zone $z \in Z$ at period $t \in T$. Notice that there is only one slice i which is selected as the cumulative investment slice for zone z and period t. This means that the variable $f^{inv}_{z,t,i}$ takes the value of $u^{inv}_{z,t}$ when $c^{tot}_{z,t,i} = 1$.

Let's now shift our focus to the capex budget constraints. Consider a committee period $t \in T^C$ and an investment slice $i \in I$. A subset of zones A in Z is said to be a cover for the CAPEX budget of committee t and slice i if the total CAPEX cost of these zones over all periods in P_t exceeds the CAPEX budget $Budget_t$. Moreover, it is said to be a minimal cover for the committee t when for each $a \in A$, the subset $A \setminus \{z\}$ does not define a cover for the committee t. This means that we should not invest in all zones A together at period t. Otherwise, the budget constraint (15) is violated. Based on this, we introduce the following so-called *cover-based* inequalities.

Proposition 16. *Consider a committee period $t \in T^C$ and a investment slice $i \in I$. Let A be a minimal cover for the CAPEX budget of committee t. Then, the following inequality*

$$\sum_{z \in A} c^{upg}_{z,t,i} \leq |A| - 1, \tag{31}$$

is valid for the polytope $\mathcal{P}(Z,T,T^C,I,Q,D,d,Budget)$.

This ensures that there is at least one zone $z \in A$ that cannot be upgraded together with other zones in A.

Remark3: notice that all the inequalities introduced in this section are also valid for the polytope $\tilde{\mathcal{P}}(Z,T,T^C,I,Q,D,d,Budget)$ associated with formulation II. The only modification required is to replace t with $C(t)$, $t+1$ with $C(t+1)$ and $t-1$ with $C(t-1)$. Additionally, when defining the conflict graph $G^z_{I,T}$, we should consider only the periods in T^C.

5 BRANCH-AND-CUT ALGORITHM

Based on the previous formulation, an exact algorithm is developed using a Branch-and-Cut approach to solve the problem. The algorithm solves a sequence of linear programs using a cutting-plane method at each node of the Branch-and-Bound (B&B) algorithm. At each iteration, the cutting-plane method generates additional inequalities (called cuts) if the current solution violates some valid inequalities. It is important to note that the cutting-plane algorithm provides an optimal solution for the linear relaxation of the problem, which may not be feasible for the original problem if it violates the integrality constraints. In such cases, the algorithm proceeds to the branching step, where the problem is divided into subproblems by branching on integer variables. This process continues until an optimal solution is obtained.

On the other hand, and as discussed before, our valid inequalities are derived from three well-known classes: cover, clique and odd-cycle inequalities. The separation problem for cover and clique inequalities (Schrijver, 1986) are well known to be NP-Hard (Nemhauser and Wolsey, 1988). (Nemhauser and Sigismondi, 1992) proposed a greedy algorithm to obtain an approximate solution for these problems. This has been adapted to provide an approximate solution for solving the separation problem of our cover and clique inequalities. However, the separation problem for odd-cycle inequalities can be solved exactly in polynomial time as shown by Grötschel et al. in (Grötschel et al., 1988). These findings remain applicable to the valid inequalities presented in this work.

6 COMPUTATIONAL STUDY

We implemented the two formulations previously described, using the Pyomo package with Python as programming language. For each MILP, we developed a Branch-and-Cut algorithm to solve the problem. To solve each MILP formulation, we relied on CPLEX 12.9, benefiting from its own cuts to obtain tighter bounds for the linear relaxation, thereby enhancing the performance of the Branch-and-Cut algorithm.

6.1 Test Setting

The tests have been run on a server with 256GB RAM and 32 threads running in parallel. The maximum CPU time has been set up to 1 hours (3600 sec).

To assess the efficiency of our approaches, we conducted experiments using different instances size with varying the number of zones, the time horizon size, and the number of committee periods. Specifically, we considered $|Z| \in \{25, 100, 250, 500\}$ zones and $|T| \in \{12, 36, 120\}$ monthly periods, with $|T^C| \in \{1, 3, 10\}$ committees (one committee per year): Each instance (size) is thus characterized by a triplet $(|Z|, |T|, |T^C|)$ and denoted as $|Z|\_|T|\_|T^C|$.

Moreover in order to simulate relevent CAPEX budgets, we first investigated reference scenarios:

- WITHOUT_UPGRADE: in this scenario, we did not allow any new co-investment during the time horizon, which means that the cumulative investment rate of each zone $z \in Z$ remains constant at $q_{z,0}$ throughout the entire horizon T. Consequently, CAPEX costs are constant and only those induced by the growth of the number of connectable customers incurr. Let $B0_{|Z|,|T|,|T^C|)_t}$ represent the total CAPEX paid for each committee $t \in T^C$ (covering all CAPEX costs incurred over all periods in P_t) in each instance $(|Z|, |T|, |T^C|)$.
- INFINITE_CAPEX : here, we did not take into account the budget constraint (15). The purpose of this was to calculate the CAPEX budget required to achieve the minimum value for the sum of the renting, opex and migration cost (16). Consequently, this value provides the lower bound for the general case when considering the budget constraint (15). Let $B1_{|Z|,|T|,|T^C|)_t}$ represent the CAPEX budget paid for each committee $t \in T^C$ in each instance $(|Z|, |T|, |T^C|)$.

Notice that the optimal solution for scenario WITHOUT_upgrade is trivial. This problem can be solved in polynomial time, specifically in $O(|Z| \cdot |T|)$. The optimal decision for each zone $z \in Z$ is as follows

- $c^{upg}_{z,t,i} = 0$ for all $i \in I$ with $Q_i > 0\%$,

- $c^{tot}_{z,t,i} = 1$ if $Q_i = q_{z,0}$ and 0 if not,
- $q_{z,t} = q_{z,0}$ and $u^{mig}_{z,t} = mig_{z,t} = 0$,
- if $RENT_{z,t} \geq SUB_{z,t}$ then $u^{inv}_{z,t} = \min(d_{z,t}, \bar{d}_{z,t})$ and $u^{rent}_{z,t} = d - \min(d_{z,t}, \bar{d}_{z,t})$,
- if $RENT_{z,t} \leq SUB_{z,t}$ then $u^{rent}_{z,t} = \min(d_{z,t}, \bar{d}_{z,t})$ and $u^{inv}_{z,t} = d - \min(d_{z,t}, \bar{d}_{z,t})$.

However, the problem becomes more complex in the INFINITE_CAPEX scenario, leading to a substantial increase in the number of potential feasible solutions. For this, the MILPI is used to solve the problem. Relying on the results of these two reference scenarios, we propose three additional ones with different CAPEX budgets to further investigate the behavior of our branch-and-cut algorithm. These scenarios are denoted by $B\alpha\%$ with $\alpha \in \{100\%, 75\%, 50\%, 25\%, 0\%\}$ where $B100\%$ (resp. $B0\%$) corresponds to scenario INFINITE_CAPEX (resp. WITHOUT_UPGRADE). For each scenario $B\alpha\%$, the CAPEX budget for each committee $t \in T^C$ is calculated as $B0_{|Z|,|T|,|T^C|)_t} + \alpha * (B1_{|Z|,|T|,|T^C|)_t} - B0_{|Z|,|T|,|T^C|)_t})$.

Table 2: MILPI Vs MILPII using PARAM_1.

Instances		MILP I			MILP II		
Bα%	Name	Nd	Gap	TT	Nd	Gap	TT
B100%	25_12_1	1	0,00	2,4	1	0,00	7,2
	25_36_3	1	0,00	9,9	1	0,00	3,1
	25_120_10	6253	0,00	234,3	1	0,00	13,1
	100_12_1	236	0,00	34,9	1	0,00	4,4
	100_36_3	5925	0,00	305,8	3	0,00	23
	100_120_10	6265	0,00	907,5	1	0,00	68,9
	250_12_1	6406	0,00	287,5	1	0,00	116,1
	250_36_3	6654	0,00	837,57	7	0,00	67,9
	500_12_1	6367	0,00	1009,2	135	0,00	134,3
	500_36_3	9310	116,2	3 600	131	0,00	194,1
B75%	25_12_1	1	0,00	4,9	1	0,00	3,5
	25_36_3	621010	0,003	3 600	2178275	0,002	3600
	25_120_10	1	0,00	1,7	1	0,00	1,2
	100_12_1	38263	0,00	92,8	14296	0,00	26,9
	100_36_3	188045	0,003	3600	1530152	0,001	3600
	100_120_10	5387	0,00	43,5	3246	0,00	19,1
	250_12_1	1480	0,00	136,8	986	0,00	72,5
	250_36_3	44798	0,002	3600	53548	0,00	3600
	500_12_1	9429	0,00	611,4	38889	0,00	982
	500_36_3	5926	0,2	3600	24416	0,8	3600
B50%	25_12_1	9	0,00	4,3	1	0,00	10,2
	25_36_3	5517	0,00	36,7	190069	0,00	125,2
	25_120_10	1	0,00	1,7	1	0,00	1,2
	100_12_1	9167	0,00	29	2655	0,00	19,8
	100_36_3	226990	0,002	3600	900624	0,001	3600
	100_120_10	1	0,00	22,7	1	0,00	11,5
	250_12_1	67917	0,00	652,6	43693	0,00	211,2
	250_36_3	126391	0,00	3600	74672	0,00	915,7
	500_12_1	36106	0,00	1 371,2	47726	0,00	1 411
	500_36_3	12005	0,01	3600	10404	0,37	3600
B25%	25_12_1	2235	0,00	7,2	1268	0,00	92,6
	25_36_3	8723	0,00	40,3	25405	0,00	21,7
	25_120_10	1	0,00	1,7	1	0,00	1,2
	100_12_1	9038	0,00	25,8	642	0,00	17,5
	100_36_3	188045	0,003	3600	1530152	0,001	3600
	100_120_10	116	0,00	19,8	29	0,00	9,4
	250_12_1	3508340	0,0001	3600	3531930	0,0001	3600
	250_36_3	70612	0,001	3600	76767	0,001	3600
	500_12_1	7496	0,00	433,5	15472	0,00	307,2
	500_36_3	17990	0,003	3600	14637	0,005	3600

Table 3: MILPI Vs MILPII using PARAM_2.

Instances		MILP I			MILP II		
Bα%	Name	Nd	Gap	TT	Nd	Gap	TT
B100%	25_12_1	1	0,00	103,8	1	0,00	270,5
	25_36_3	1	0,00	2,2	1	0,00	1,7
	25_120_10	1	0,00	8,3	1	0,00	4,3
	100_12_1	1	0,00	3,8	1	0,00	1,7
	100_36_3	1	0,00	8,8	1	0,00	5,3
	100_120_10	1	0,00	37,7	1	0,00	23,1
	250_12_1	1	0,00	9,3	1	0,00	4,6
	250_36_3	1	0,00	30,8	1	0,00	14,9
	500_12_1	1	0,00	24,4	1	0,00	10,4
	500_36_3	1	0,00	79,7	3	0,00	40,5
B75%	25_12_1	1	0,00	292,9	1	0,00	0,8
	25_36_3	105355	0,00	177,1	149396	0,00	86,7
	25_120_10	1	0,00	1,7	1	0,00	1,2
	100_12_1	1178	0,00	11,9	35	0,00	4,7
	100_36_3	12042	0,00	155,2	18610	0,00	82,8
	100_120_10	1	0,00	13,1	1	0,00	7,4
	250_12_1	1005	0,00	45,7	312	0,00	14,2
	250_36_3	6575	0,00	556,7	5432	0,00	97,3
	500_12_1	6650	0,00	329,2	5671	0,00	98,8
	500_36_3	50874	0,07	3600	75729	0,06	3600
B50%	25_12_1	1	0,00	290,1	1	0,00	0,7
	25_36_3	73738	0,00	180,2	67812	0,00	33,6
	25_120_10	1	0,00	1,7	1	0,00	1,2
	100_12_1	61	0,00	11,1	46	0,00	4,1
	100_36_3	20017	0,00	235,9	12837	0,00	84,9
	100_120_10	13	0,00	16,7	1	0,00	7,4
	250_12_1	5459	0,00	106,7	5430	0,00	33,4
	250_36_3	8352	0,00	224,8	617	0,00	35,2
	500_12_1	5609	0,00	258,4	362291	0,00	1 970,7
	500_36_3	40163	0,00	2 868,1	43120	0,00	1 990,3
B25%	25_12_1	1159	0,00	68,6	761	0,00	41,8
	25_36_3	8903	0,00	28,3	8132	0,00	7,3
	25_120_10	1	0,00	1,7	1	0,00	1,2
	100_12_1	1	0,00	4,5	5	0,00	3,9
	100_36_3	25023	0,00	233,9	8548	0,00	57,5
	100_120_10	1	0,00	11,2	1	0,00	7,2
	250_12_1	6416	0,00	95,3	319	0,00	13,5
	250_36_3	6301	0,00	267,3	9614	0,00	118,3
	500_12_1	6360	0,00	215,3	13762	0,00	104,3
	500_36_3	5652	0,00	754,2	29215	0,00	2390,9

6.2 Numerical Results

In Tables 2-5, we present a comparison between our two MILP using different parameterizations related to the values of $q_{z,0}$ and $\bar{q}_z$ for zone Z. The parameterizations are defined as follows:

- PARAM_1: $q_{z,0} = 0\%$ and maximum investment rate $\bar{q}_z$ is unbounded for all zones,
- PARAM_2: $q_{z,0} = 0$ and $\bar{q}_z$ is bounded for all zones, as specified in inequalities ($\tilde{4}$),
- PARAM_3: $q_{z,0} \geq 5\%$ for certain zones and $\bar{q}_z$ is bounded for all zones, as indicated in inequalities ($\tilde{4}$),
- PARAM_4: $q_{z,0} \geq 5\%$ for certain zones, with $\bar{q}_z = Q^{max}_z$ for all zones.

We report 3 criteria in the computational study:

- the number of nodes in the B&C tree (Nd),
- the optimality gap (Gap), given in %, which represents the relative error between the lower bound and best upper bound obtained at the end of the resolution,
- the total CPU time computation (TT), given in seconds.

Table 4: MILPI Vs MILPII using PARAM_3.

Instances		MILP I			MILP II		
Bα%	Name	Nd	Gap	TT	Nd	Gap	TT
B100%	25_12_1	1	0,00	0,2	1	0,00	0,2
	25_36_3	1	0,00	0,9	1	0,00	0,5
	25_120_10	1	0,00	2,7	1	0,00	1,5
	100_12_1	1	0,00	1,2	1	0,00	1,2
	100_36_3	1	0,00	3,6	1	0,00	1,9
	100_120_10	1	0,00		1	0,00	7,6
	250_12_1	1	0,00	2,9	1	0,00	1,6
	250_36_3	1	0,00	9,9	1	0,00	5,5
	500_12_1	1	0,00	6,2	1	0,00	3,3
	500_36_3	1	0,00	17,9	1	0,00	11,1
B75%	25_12_1	1	0,00	0,2	1	0,00	0,2
	25_36_3	1	0,00	0,9	1	0,00	0,5
	25_120_10	1	0,00	3,5	1	0,00	3,2
	100_12_1	1	0,00	1,2	1	0,00	0,6
	100_36_3	1	0,00	3,1	1	0,00	1,8
	250_12_1	1	0,00	2,9	1	0,00	1,6
	250_36_3	1	0,00	10,3	1	0,00	5,6
	500_12_1	1	0,00	6,2	1	0,00	3,2
	500_36_3	1	0,00	20,7	1	0,00	11
B50%	25_12_1	1	0,00	0,2	1	0,00	0,2
	25_36_3	1	0,00	0,9	1	0,00	0,5
	25_120_10	1	0,00	3,3	1	0,00	1,5
	100_12_1	1	0,00	1,2	1	0,00	0,6
	100_36_3	1	0,00	3,6	1	0,00	1,9
	250_12_1	1	0,00	2,7	1	0,00	1,6
	250_36_3	1	0,00	10,1	1	0,00	5,6
	500_12_1	1	0,00	6,4	1	0,00	3,2
	500_36_3	1	0,00	20,7	1	0,00	10,5
B25%	25_12_1	1	0,00	0,2	1	0,00	0,2
	25_36_3	1	0,00	0,9	1	0,00	0,5
	25_120_10	1	0,00	3,5	1	0,00	1,5
	100_12_1	1	0,00	1,2	1	0,00	0,6
	100_36_3	1	0,00	3,6	1	0,00	1,9
	250_12_1	1	0,00	2,9	1	0,00	1,6
	250_36_3	1	0,00	9,9	1	0,00	5,1
	500_12_1	1	0,00	6,3	1	0,00	3,2
	500_36_3	1	0,00	20,5	1	0,00	11

Table 5: MILPI Vs MILPII using PARAM_4.

Instances		MILP I			MILP II		
Bα%	Name	Nd	Gap	TT	Nd	Gap	TT
B100%	25_12_1	1	0,00	0,3	1	0,00	0,2
	25_36_3	1	0,00	0,5	1	0,00	0,5
	25_120_10	1	0,00	1,9	1	0,00	1,8
	100_12_1	1	0,00	1,3	1	0,00	0,9
	100_36_3	1	0,00	3,2	1	0,00	2,1
	250_12_1	1	0,00	2,9	1	0,00	1,7
	250_36_3	1	0,00	6,2	1	0,00	5,7
	500_12_1	1	0,00	6,2	1	0,00	3,5
	500_36_3	1	0,00	12,2	1	0,00	11,7
B75%	25_12_1	1	0,00	0,3	1	0,00	0,2
	25_36_3	1	0,00	1	1	0,00	0,5
	25_120_10	1	0,00	2,6	1	0,00	2,1
	100_12_1	1	0,00	1,3	1	0,00	0,8
	100_36_3	1	0,00	2,6	1	0,00	2
	250_12_1	1	0,00	3	1	0,00	1,7
	250_36_3	1	0,00	6,3	1	0,00	5,8
	500_12_1	1	0,00	6,3	1	0,00	3,5
	500_36_3	1	0,00	11,8	1	0,00	11,6
B50%	25_12_1	1	0,00	0,3	1	0,00	0,2
	25_36_3	1	0,00	0,9	1	0,00	0,5
	25_120_10	1	0,00	2,4	1	0,00	1,9
	100_12_1	1	0,00	1,1	1	0,00	1
	100_36_3	1	0,00	3,4	1	0,00	2
	250_12_1	1	0,00	3	1	0,00	1,7
	250_36_3	1	0,00	6	1	0,00	5,6
	500_12_1	1	0,00	6,3	1	0,00	3,5
	500_36_3	1	0,00	11,9	1	0,00	11,6
B25%	25_12_1	1	0,00	0,3	1	0,00	0,2
	25_36_3	1	0,00	0,9	1	0,00	0,5
	25_120_10	1	0,00	2,4	1	0,00	1,9
	100_12_1	1	0,00	1,1	1	0,00	0,9
	100_36_3	1	0,00	3,5	1	0,00	2,1
	250_12_1	1	0,00	2,9	1	0,00	1,7
	250_36_3	1	0,00	5,9	1	0,00	5,6
	500_12_1	1	0,00	6,4	1	0,00	3,5
	500_36_3	1	0,00	12,4	1	0,00	11,6

The results indicate that our two mixed integer linear programs exhibit excellent performance, successfully solving nearly all instances to optimality within just a few minutes. For this, 94.1% (resp. 92.2%) of instances are solved to optimality by the second (resp. first) formulation. Furthermore, the second formulation yields better results for 50% of instances that remain unsolved to optimality by both formulations. It also successfully solves several instances to optimality that the first formulation does not. Regarding the CPU time computation, the second formulation significantly outperforms the first, achieving solutions in shorter CPU times for 85.62% of instances. We observe also that 68.62% (resp. 40.10%) of instances are solved to optimality in under 15 seconds by the second (resp. first) formulation. On the other hand, the branching tree associated with the Branch-and-Cut algorithm when using the second formulation shows a reduced number of nodes for 24.18% instances compared to the first formulation. Also, the second formulation solves 3.27% of instances to optimality at the root of the branching tree, while the first formulation requires more nodes for the same instances (i.e., branching is required to achieve optimal solutions). This clearly shows the advantages of the second formulation in effectively solving the problem. Additionally, we observed that the problem becomes increasingly complex to solve to optimality when the parameter $q_{z,0}$ is set to zero, and the inequalities in ($\tilde{4}$) are relaxed for all zones in Z. This complexity arises from the combinatorial nature of the problem, resulting in a significant increase in the number of potential feasible solutions.

Since nearly all instances have been solved to optimality, we cannot assess the impact of our additional valid inequalities on the B&C algorithm's effectiveness. This limits our evaluation of their influence on solution times and overall performance. We need to generate more complex instances that challenge optimal solutions with our formulations. Further computational studies are essential to determine how our valid inequalities can accelerate solution times for instances solved to optimality without them.

7 CONCLUSION

In this paper, we have addressed the strategic decision problem of a telecommunication operator which aims at efficiently planning its future investments in Fiber access of geographical areas that are currently undeployed. We have introduced two mixed integer linear programs to model the problem. Additionally, we presented several classes of valid inequalities for the associated polytopes. These results are used to develop a Branch-and-Cut algorithm for solving the problem. An extensive computational study was conducted to assess the algorithm's performance across various instances and scenarios. While our approach has proven effective, it would be beneficial to investigate the impact of our valid inequalities in the context of larger and more complex instances. Such an analysis could yield valuable insights into the strengths and weaknesses of both approaches.

ACKNOWLEDGEMENTS

We would like to acknowledge the contribution of the Orange Wholesale France team members who have contributed to this study and preparation of the real data for our computational study. We are grateful for their expertise in making this work possible.

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