A SAMPLING FORMULA FOR DISTRIBUTIONS

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Abstract: A key sampling formula for discretising a continuos-time system is proved when the signals space is a subclass of the space of Distributions. The result is applied to the analysis of an open-loop hybrid system.

1 INTRODUCTION

Consider the hybrid system of Figure 1, where $x(t)$ and $y(t)$ are input and output, $(A/D)_T$ is an A/D converter with sampling period T, $(D/A)_T$ is a zero-order hold (ZOH) and P and C are the plants of a continuous time system and a discrete time system, respectively. In order to perform the transform domain analysis of the hybrid system of Figure 1, the transform domain response of a sampled signal must be related to the transform response of its correspondent continuous time signal. This is done by building the transform response of the sampled signal upon the superposition of infinitely many copies of its continuous time transform response, using the formula

$$G_d(e^{st}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G(s+jk\omega_s) \tag{1}$$

where G is the Laplace transform of a continuous time signal g, G_d is the z transform of the sequence of its samples $\{g(kT)\}_{k=0}^{\infty}$ and T and $\omega_S = 2\pi/T$ are the sampling period and the sampling frequency, respectively.

Till 1997, with the publication of (Braslavsky et al., 1997), 1 was mathematical folklore. In fact, it was very often used in the digital control literature ((M.Araki and T.Hagiwara, 1996), (J.S.Freudenberg and J.H.Braslavsky, 1995), (T.Hagiwara and M.Araki, 1995)), (Leung et al., 1991), (Y.N.Rosenvasser, 1995a), (Y.N.Rosenvasser, 1995b) and (Yamamoto and Araki, 1994)) and it appeared in many control textbooks ((K.J.Astrom and B.Wittenmark, 1990), (T.Chen and B.A.Francis, 1995), (G.F.Franklin and M.L.Workman, 1990)), (B.C.Kuo, 1992) and (K.Ogata, 1987)), but it was not established by a rigorous proof that indicated the relevant classes of signals considered.

Attempts to provide 1 with a proof are in (E.I.Jury, 1958), (K.J.Astrom and B.Wittenmark, 1990) and (T.Chen and B.A.Francis, 1995). Those proofs are based on the use of impulse trains of impulse trains, those defined as the function

$$\sum_{k=-\infty}^{\infty} \delta(x-nT)$$

where $\delta(x)$ is the impulse function or Dirac function or Dirac impulse such that

$$\delta(x) = \begin{cases} +\infty & x=0 \\ 0 & otherwise \end{cases}$$

and

$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$

However, the proofs lack rigour, since the impulse function, and hence the impulse trains, cannot be defined as functions.

In (J.R.Ragazzini and G.F.Franklin, 1958) it is shown the similarity between 1 and the Poisson Summation Formula

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} f(s)e^{-2\pi iks}ds$$

Consequently, 1 is often indicated as the Poisson Sampling Formula. In (G.Doetsch, 1971) a rigorous

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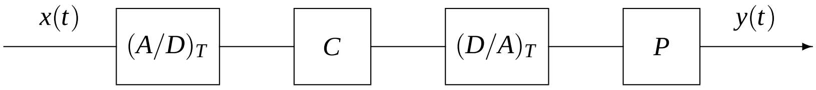

Figure 1: Open Loop Hybrid System.

proof,that avoids the use of the impulse trains, for

$$G_d(e^{st}) = \frac{g(0^+)}{2} + \frac{1}{T}\sum_{k=-\infty}^{\infty} G(s+jk\omega_s)$$

is derived under the assumption that the series $\sum_k G(s+jk\omega_s)$ is uniformly convergent. However, since this condition is a transform domain condition, it is not obvious when a time domain function satisfies it.

In (Braslavsky et al., 1997) it is pointed that for 1 to hold, it is not enough to require that the Laplace transform G of g and its sampled version, G_D, are well defined. It is shown that, for $n_p = 2^{2^{2^p}}$ and the continuous function

$$g(t) = sin((2n_p+1)t),\ t \in [p\pi,(p+1)p],\ p \in \mathbb{N}$$

1 does not hold, despite the fact that $G_d(e^{st})$ and its sampled version with period $T = \pi$, are both well defined in the open right-half plane. In fact, it is proved that

$$\lim_{n=\infty} \sum_{k=-n}^{n} G(s+jk\omega_s)$$

does not converges for any $s \geq 0$. Because of the rapid oscillations of g as $t \to \infty$ the class of signals is restricted to functions with bounded and uniform bounded variation.

Definition 1 ((Braslavsky et al., 1997)). A function g defined on the closed real interval $[a,b]$ is of bounded variation (BV) when the total variation of g on $[a,b]$,

$$V_g(a,b) = \sup_{a=t_0<t_1<\ldots<t_{n-1}<t_n=b} \sum_{k=1}^{n} |g(t_k) - g(t_{k-1})|$$

is finite. The supremum is taken over every $n \in \mathbb{N}$ and every partition of the interval $[a,b]$ into subintervals $[t_k, T_{k+1}]$ where $k = 0,1,\ldots,n-1$ and $a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b$.

A function g defined on the positive real axis is of uniform bounded variation (UBV) if for some $\Delta > 0$ the total variation $V_g(x, x+\Delta)$ on intervals $[x, x+\Delta]$ of length Δ is uniformly bounded, that is, if

$$\sup_{x\in\mathbb{R}_0^-} V_g(x, x+\Delta) < \infty$$

With the class of signals restricted to UBV functions, a proof for

$$G_d(e^{st}) = \frac{g(0^+)}{2} + \sum_{k=1}^{\infty} \frac{g(kT^+) - g(kT^-)}{2} e^{-skT} + \frac{1}{T}\sum_{k=-\infty}^{\infty} G(s+jl\omega_s)$$

a more general formulation of 1, is provided.

Note that the well posedness of 1 is proved for an open loop context, when the system considered is stable. Despite the fact that it is rather common to analyse a hybrid feedback system with the help of 1, even if the class of signals is restricted to UBV functions, there is no proof of the well posedness of the feedback when applying 1.

The discussion about the consistency of Mathematical Frameworks in Systems Theory that started with the exposure of the Georgiou Smith paradox in (Georgiou and Smith, 1995) made Leithead and al., in (Leithhead and J.O'Reilly, 2003) and (W.E.Leithead et al., 2005), to attempt a Mathematical Framework that expands the class of signals to the class of Distributions (an advantage of a Framework using Distributions is that signals like steps, train pulses and delta functions can be rigorously defined as distributions). Consequently, when dealing with hybrid systems, as the one of Figure 1, in a Distributions Framework, the well posedeness of 1 must be proved again.

However, despite 1 being quoted in Theorem 16.8 of (D.C.Champeney, 1987), no proof could be found in the literature. In this paper a rigorous proof of Theorem 16.8 of (D.C.Champeney, 1987), establishing 1 in a Distributions context, is provided in the Appendix. Furthermore, an application of this result to a open loop hybrid system is provided. In particular, a correct formulation for the D/A and A/D converters in a Distributions context is established.

2 SAMPLING THE TRANSFORMS OF A DISTRIBUTION

The following notations and conventions are adopted.

The value assigned to each $\phi(t) \in D$, the class of good functions with finite support, by the functional $x \in \mathcal{D}$, the class of distributions , is denoted by $x[\phi(t)]$. The symbols for, respectively a regular functional in $\mathcal{D}$ and the ordinary function by which it is defined, e.g. x and $x(t)$, are distinguished by the explicit presence in the latter of the variable. The following subclasses of $\mathcal{D}$ are required.

$\mathcal{D}_B =$ $\{x \in \mathcal{D} : x$ regular with $x(t)$ BV on each finite interval and $|x(t)| \leq c(1+|t|)^N$ for some $c > 0\}; N \geq 0$

$\mathcal{D}_{BN} =$ $\{x \in \mathcal{D} : x$ regular with $x(t)$ BV on each finite interval and $|x(t)| \leq c(1+|t|)^N$ for some $N \geq 0$ and $c > 0\}$

$\mathcal{D}_V =$ $\{x \in \mathcal{D} : x$ regular with $Var_{[a+t,b+t]}\{x(t)\} \leq c(1+|t|)^N$ for each finite interval $[a,b]$ for some $N \geq 0$ and $c > 0\}$

$\mathcal{D}_{VN} =$ $\{x \in \mathcal{D} : x$ regular with $Var_{[a+t,b+t]}\{x(t)\} \leq c(1+|t|)^N$ for each finite interval $[a,b]$ for some $c > 0\}; N \geq 0$

$\mathcal{D}^T =$ $\{x \in \mathcal{D} : x = \sum_{-\infty}^{\infty} a_k \delta_{kT}\}; T > 0$

$\mathcal{D}_B^T =$ $\{x \in \mathcal{D} : x = \sum_{-\infty}^{\infty} a_k \delta_{kT}$ with $|a_k| \leq (1+|k|)^N$ for some $c > 0$ and $N \geq 0\}; T > 0$

$\mathcal{D}_{BN}^T =$ $\{x \in \mathcal{D} : x = \sum_{-\infty}^{\infty} a_k \delta_{kT}$ with $|a_k| \leq (1+|k|)^N$ for some $c > 0\}$; $N \geq 0, T > 0$

where $Var_{[a,b]}\{x(t)\}$ is the variation of $x(t)$ on the interval $[a,b]$ and the functional δ_τ is the delta functional in $\mathcal{D}$ defined by

$$\delta_\tau[\phi(t)] = \phi(\tau)$$

Each functional $x \in \mathcal{D}$ is related by a linear bijections to a functional $\mathcal{U}$ such that

$$x[\phi(t)] = 2\pi X[\Phi(\omega)]$$

for all $\phi(t) \in D$ with

$$\Phi(\omega) = \mathcal{F}[\phi(t)](\omega)$$

The functionals x and X constitutes a Fourier transform pair with

$$X = \mathcal{F}\{x\} \text{ and } x = \mathcal{F}^{-1}\{X\}$$

The subclasses $\mathcal{U}_B$, $\mathcal{U}_{BN}$, $\mathcal{U}_V$, $\mathcal{U}_{VN}$, $\mathcal{U}_B^T$ and $\mathcal{U}_{BN}^T$ are the Fourier transforms of the the corresponding subclass of $\mathcal{D}$. The members of $\mathcal{U}^T$ and its subclasses are periodic with period $2\pi/T$.

A multiplier in $\mathcal{D}$ is an ordinary function $f(x)$ that is infinitely differentiable at each real value of x. The multipliers in $\mathcal{D}$ are denoted by $\mathcal{M}$. The subclass $\mathcal{M}^T$ is the class of periodic multipliers with period $2\pi/T$.

The relations between the transform of a distribution and its sampled version is established in the following Theorem.

Theorem 2 (16.8 (D.C.Champeney, 1987))**.** *Suppose $\tilde{f} \in \mathcal{U}$ has a transform $\tilde{F} \in \mathcal{D}$ that is regular and equal to a function F that is of bounded variation on each finite interval (though not necessarily on $(-\infty,\infty)$): then*

(i) $F(y)$ will be equal a.e. to a function $F_D(y)$ such that, at all y,

$$F_D(Y) = \frac{1}{2}[F_D(y^-) + F_D(y^+)]$$

(ii) also

$$X \sum_{-\infty}^{\infty} \tilde{f}(x - nX) \tag{2}$$

will converge in $\mathcal{U}$ to define a periodic functional $\tilde{g}$ whose Fourier coefficients G_n are given by

$$G_n = F_D(n/X),\ n = 0, \pm1, \pm2, \ldots$$

(iii) if in addition $\tilde{f} \in \mathcal{D}_S$ and $F(y)/(1+|y|)^N$ is of bounded variation on $(-\infty,\infty)$, then 2 will converge in $\mathcal{D}_S$.

A proof of 2 is given in the Appendix.

3 OPEN LOOP HYBRID FEEDBACK SYSTEM

Reconsider the plants P and C of the open loop hybrid system of Figure 1 as the stable systems on $\mathcal{D}_E$ and $\mathcal{D}_E^T$, respectively.

$$C : x \in \mathcal{D}^T \mapsto y \in \mathcal{D}^T, y = \Psi * x$$

$$P : x \in \mathcal{D} \mapsto y \in \mathcal{D}, y = \Phi * x$$

where Ψ and Φ are convolutes on $\mathcal{D}^T$ and $\mathcal{D}$, respectively. However, since it is required that the idealised sampling of continuous time signal is well-defined, a more appropriate reformulation of continuous time signals is provided by the subclass of distributions $\mathcal{D}_B$.

Consequently, the convolutes Ψ and Φ corresponding to plants C and P must be restricted to $\mathcal{D}_B^T$ and $\mathcal{D}_B$, respectively. In transform domain the Fourier transforms of signals are represented by functionals in $\mathcal{U}_B$ and the transfer functions of systems are functionals in $\mathcal{M}_B$, the class of multipliers on $\mathcal{U}_B$ mapping $\mathcal{U}_{BN}$ into itself for all $N \geq 0$. It remains to establish a correct formulation of the D/A and A/D converters.

3.1 Frequency Domain Analysis - D/A Converter

Consider an ideal D/A converter which acts, with a time constant T, on a discrete time signal, $\{x[k]\}$ to produce a piecewise constant continuous time signal, $y(t)$; that is, it acts as an ideal zero-order-hold (ZOH). The linear relationship between $\{x[k]\}$ and $y(t)$ in the frequency domain is established by the following Theorem.

Theorem 3. *A discrete time signals $\{x[k]\}$ is acted on by a ZOH, with time constant T, to produce a piecewise constant time signal $y(t)$ such that*

$$y(t) = \sum_{k=-\infty}^{\infty} x[k]h^T(t-k)$$

where $h^T(t) = 1$ when $t \in [0,T)$, zero otherwise. Provided there exists a periodic functional $X \in \mathcal{U}_{BN}^T$ with Fourier coefficients $\{x[k]\}$, then $y(t)$ defines a regular functional, $y \in \mathcal{D}_{BN} \cap \mathcal{D}_{VN}$ such that $Y = H^T X$ where $Y = \mathcal{F}\{y\} \in \mathcal{U}_{BN} \cap \mathcal{U}_{VN}$ and $H^T = \mathcal{F}\{h^T\}$ with h^T the functional in $\mathcal{D}$ defined by $h^T(t)$.

Proof. $y(t)$ is of bounded variation on any finite interval, and, since $X \in \mathcal{U}_{BN}^T$ implies $|x[k]| \leq c(1+|k|)^N$ for some c, $|y(t)| < c^*(1+|t|)^N$ for some c^*. Hence $y = \sum_{k=-\infty}^{\infty} x[k]h_{kT}^T \in \mathcal{D}_{BN}$. Furthermore for all $b_i \in \{-1,1\}$ and $\{\tau_1, \tau_2, \ldots, \tau_{n+1}\}$ satisfying $a \leq \tau_1 < \tau_2 < \ldots < \tau_{n+1} \leq b$

$$\begin{aligned}
&\sum_{i=1}^{n} b_i(y(t+\tau_{i+1}) - y(t+\tau_i)) \\
&\quad = \sum_{i=1}^{\bar{n}} b_i(y(t+\tau_{i+1}) - y(t+\tau_i)) \\
&\quad \leq \sum_{i=1}^{\bar{n}} (|y(t+\tau_{i+1})| + |y(t+\tau_i)| \\
&\quad \leq \sum_{i=1}^{\bar{n}} (c^*(1+|t+\tau_{i+1}|) + c^*(|t+\tau_i|) \\
&\quad \leq 2c^*\bar{n}(1+|t+b|)^N
\end{aligned}$$

where $\bar{n} = int(t/(kT))$. Hence, $Var_{[a+t,b+t]}\{y(t)\} \leq \bar{c}(1+|t|)^N$, for some $\bar{c} > 0$, and $y \in \mathcal{D}_{VN}$. In addition, since h^T is a convolute on $\mathcal{D}$,

$$\begin{aligned}
y &= \lim_{n\to\infty} h^T * \sum_{k=-n}^{n} x[n]h_{kT}^T \\
&= \lim_{n\to\infty} * \sum_{k=-n}^{n} x[k]\delta_{kT} \\
&= h^T * \lim_{n\to\infty} \sum_{k=-n}^{n} x[k]\delta_{kT} = h^T * x
\end{aligned}$$

with $x = \mathcal{F}^{-1}\{X\}$ and $Y = H^T X$ as required. ■

Therefore, a D/A converter is represented in the frequency domain by the multiplier H^T mapping $\mathcal{U}_{BN}^T$ into $\mathcal{U}_{BN} \cap \mathcal{U}_{VN}$. Moreover, as a consequence, a discrete time subsystem positioned before a D/A converter is equivalent to a continuous time subsystem positioned after the D/A converter, provided their frequency response functions are the same.

3.2 Frequency Domain Analysis - A/D Converter

Consider an ideal A/D converter which samples, with a sampling interval T, a continuous time signal, $x(t)$, to produce a discrete time signal $\{y[k]\} = \{x[k]\}$. The linear relationship between $x(t)$ and $\{y[k]\}$ in the frequency domain is established by the following Theorem.

Theorem 4. *A continuous time signal, $x(t)$, is acted by a sampler with sampling interval T to produce a discrete time signal $\{y[k]\}$. Provided there exists a regular functional $x \in \mathcal{D}_{BN}$ defined by $x(t)$ then*

(i) $x(t)$ is equal almost everywhere to a function $x_D(t)$ such that, at all t,

$$x_D(t) = \frac{(x_D^-(t) + x_D^+(t))}{2}$$

and so sampling is well defined with $y[k] = x_D(kT)$.

(ii) the summation $\frac{1}{T}\sum_{k=-\infty}^{\infty} X_{2\pi k/T}$ converges in $\mathcal{U}$, where $X = \mathcal{F}\{x\} \in \mathcal{U}_{BN}$, and $\{y[k]\}$ are the Fourier coefficients for a periodic functional $Y \in \mathcal{U}_{BN}^T$ with period $2\pi/T$ such that $Y = \mathcal{O}^T[X] = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_{2\pi/T}$

Proof. Since $X \in \mathcal{D}_{BN}$, $x(t)$ is of bounded variation on each finite interval and part (i) follows from Theorem 2. In addition, there exists a periodic functional $Y \in \mathcal{U}$, with period $2\pi/T$ and Fourier coefficients $y_k[k] = x_D(kT)$ such that the summation $\frac{1}{T}\sum_{k=-\infty}^{\infty} X_{2\pi k/T}$ converges in $\mathcal{U}$ and $Y = \mathcal{O}^T[X] = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_{2\pi k/T}$. Furthermore, since $x \in \mathcal{D}_{BN}$, $y = \mathcal{F}^{-1}\{Y\} \in \mathcal{D}_{BN}^T$ as required by part (ii). ■

Therefore, an A/D converter is represented in the frequency domain by the linear operator $\mathcal{O}^T$ on $\mathcal{U}_B$ mapping $\mathcal{U}_{BN}$ into $\mathcal{U}_{BN}^T$ for all $N \geq 0$. Further properties of the operator $\mathcal{O}^T$ are established in the following Theorem.

Theorem 5. *If X is a functional in $\mathcal{U}_B$ with n^{th} derivative $X^{(n)}$, Y is a functional in $\mathcal{U}_B$ and M^T is a periodic multiplier in $\mathcal{M}_B$ with period $2\pi/T$ then*

(i) $\mathcal{O}^T[X]$ is a periodic multiplier in $\mathcal{M}_B$ with period $2\pi/T$ provided $j^n X^{(n)} \in \mathcal{U}_{B0}$ for all $n \geq 0$;

(ii)$\mathcal{O}^T[M^T X] = M^T \mathcal{O}^T[X]$;

(iii) $\mathcal{O}^T[Y \mathcal{O}^T[X]] = \mathcal{O}^T[Y]\mathcal{O}^T[X]$ provided $j^n X^{(n)} \in \mathcal{U}_0$ for all $n \geq 0$.

Proof. (i)The regular functional $x = \mathcal{F}^{-1}\{X\} \in \mathcal{D}_B$ is defined by a function $x(t)$, which by Theorem 4 part (i) is equal almost everywhere to a function $x_D(t)$ such that, at all t,

$$x_D(t) = \frac{(x_D^-(t) + x_D^+(t))}{2}$$

For all $n \geq 0$, since $j^n X^{(n)} \in \mathcal{U}_{B0}$, $y \in \mathcal{D}_{B0}$, where y is the functional defined by $t^n x(t)$, and the series $\sum_{k=-\infty}^{\infty}(kT)^n x_D(kT)e^{-jk\omega T}$ converges for all ω. Hence, by Theorem 4 part (ii), $\mathcal{O}^T[X]$ is an infinitely differentiable regular functional. Furthermore, the n^{th} derivative of $\mathcal{O}^T[X]$ is continuous and periodic and so bounded. Consequently, $\mathcal{O}^T[X]$ is a multiplier in $\mathcal{M}_B$ with period $2\pi/T$.

(ii) For any $X \in \mathcal{U}_{BN}$, $M^T X \in \mathcal{U}_{BN}$ and by Theorem 4 both $\mathcal{O}^T[X] \in \mathcal{U}_{BN}^T$ and $\mathcal{O}^T[M^T X] \in \mathcal{U}_{BN}^T$ exist. Moreover, since M^T is a multiplier in $\mathcal{M}_B$ with period $2\pi/T$,

$$\begin{aligned}
&\frac{1}{T}\lim_{n\to\infty}\sum_{k=-n}^{n} M_{kT}^T X_{kT} \\
&\quad = \frac{1}{T}\lim_{n\to\infty}\sum_{k=-n}^{n} M^T X_{kT} = \frac{1}{T}\lim_{n\to\infty} M^T \sum_{k=-n}^{n} X_{kT} \\
&\quad = M^T \frac{1}{T}\lim_{n\to\infty}\sum_{k=-n}^{n} X_{kT}
\end{aligned}$$

and $\mathcal{O}^T[M^T X] = M^T \mathcal{O}^T[X]$ as required.

(iii) It follows directly from part (i) and (ii). ■

A consequence of Theorem 4 part (ii) is that, in frequency domain, a continuous time sub systems positioned before an A/D converter is equivalent to a discrete time subsystem positioned after the A/D provided their frequency response functions are the same.

3.3 The Response of the Open Loop Hybrid Feedback System

In time domain the stable hybrid feedback system of Figure 1 has solution

$$y = \Phi * [(D/A)_T(\Psi * [(A/D)_T x])] \qquad (3)$$

Define K_C^T and K_P the multipliers in $\mathcal{M}_B$, the transfer functions of the convolutes Ψ and Φ, respectively. Therefore, by Theorems 3 4 and 5, in Frequency Domain, to 3 corresponds the solution

$$Y = K_P[H^T(K_C[\mathcal{O}^T X])]$$

where Y and X are functionals in $\mathcal{U}_B$, the Fourier transforms of y and x.

4 CONCLUSION

In this paper the proof of the well posedness of the sampling of a the transform of a distribution is given, establishing the correctness of the Sampling Theorem 16.8 quoted in (D.C.Champeney, 1987). Moreover, the result is applied to the frequency domain response of an open loop hybrid system, through the correct formulation for the D/A and A/D converters.

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APPENDIX

Theorem 2 (D.C.Champeney, 1987)

Proof. (i) and (ii) Let $\tilde{f}_N \in \mathcal{D}$ be the regular functional defined by $f_N(x)$ where

$$f_N(x) = \sum_{n=-N}^{N} e^{jn(2\pi/X)x} = \frac{sin(\pi(2N+1)x/(2X))}{sin(\pi x/(2X))}$$

$\tilde{f}_N$ is a multiplier on $\mathcal{D}$ and $f_N(x)$ is periodic with period X such that

$$\int_{-X/2}^{X/2} f_N(x)dx = X$$

For any regular $\tilde{g} \in \mathcal{D}$, with $g(x)$ of bounded variation on any finite interval, and any $\psi(x) \in D$,

$$(\tilde{f}_N\tilde{g})[\psi(x)] = \tilde{g}[f_N(x)\psi(x)] = \int_{-\infty}^{\infty} g(x)f_N(x)\Psi(x)dx$$

Since $\psi(x)$ is of finite support, $\exists K$ such that $\psi(x) = 0$ for $|x| > (K+\frac{1}{2})X$. Hence,

$$\begin{aligned}
\tilde{f}_N\tilde{g}[\psi[x]] &= \int_{-(K+1/2)X}^{(K+1/2)X} g(x)f_N(x)\psi(x)dx \\
&= \int_{-X/2}^{X/2} \left\{ \sum_{k=-K}^{K} f_N(x)g(x+kX)\psi(x+kX) \right\} dx \\
&= \int_{-X/2}^{X/2} f_N(x)\phi_K(x)dx \\
&= \int_{-X/2}^{X/2} \left(\frac{sin\left(\frac{\pi(2N+1)x}{2X}\right)}{x} \right) \left\{ \frac{\phi_k(x)x}{sin\left(\frac{\pi x}{2X}\right)} \right\} dx
\end{aligned}$$

where

$$\phi_k(x) = \sum_{k=-K}^{K} g(x+kX)\psi(x+kX)$$

For all k, $g(x)$ is of finite variation on $[(k-1/2)X, (k+1/2)X]$ and so $\phi_K(x)x/(sin(\pi x/(2X)))$ is of finite variation on $[(k-1/2)X, (k+1/2)X]$. Consequently, by Theorem 5.10 of (D.C.Champeney, 1987), $x = 0$ is a Dirichlet point and

$$\lim_{N\to\infty} \int_{-X/2}^{X/2} (sin(\pi(2N+1)x/(2X))/x)$$
$$\{\phi_k(x)x/sin(\pi x/(2X))\}dx = X(\phi_k(0^+)+\phi_k(0^-))/2$$

It follows that

$$\begin{aligned}
&\lim_{N\to\infty} (\tilde{f}_N\tilde{g})[\psi(x)] \\
&= X\sum_{k=-K}^{K} \frac{1}{2}(g(kX^-)+g(kX^+))\psi(kX) \\
&= X\sum_{k=-K}^{K} \frac{1}{2}(g(kX^-)+g(kX^+))\tilde{\delta}_{kX}[\psi(x)]
\end{aligned}$$

Hence, $\frac{1}{N}\tilde{f}_N\tilde{g}$ converges to

$$\tilde{h} = \sum_{k=-K}^{K} \frac{1}{2}(g(kX^-)+g(kX^+))\tilde{\delta}_{kX}$$

in $\mathcal{D}$. Furthermore,

$$\tilde{H} = \mathcal{F}\{\tilde{h}\} = \sum_{k=-\infty}^{\infty} \frac{1}{2}(g(kX^-)+g(kX^+))\tilde{e}_{k(2\pi/X)} \in \mathcal{U}$$

and by Theorem 16.3 of (D.C.Champeney, 1987), $\tilde{H}$ is periodic with period $2\pi/X$ and Fourier coefficients $\left\{\frac{1}{2}(g(kX^-)+g(kX^+))\right\}$. However

$$\begin{aligned}
\mathcal{F}\left\{\frac{1}{X}\tilde{f}_N\tilde{g}\right\} &= \frac{1}{X}\mathcal{F}\{\tilde{f}_N\} * \mathcal{F}\{\tilde{g}\} \\
&= \frac{1}{X}\left(\sum_{n=-N}^{N}\tilde{\delta}_{n(2\pi/X)}\right) * \tilde{G} = \frac{1}{N}\sum_{n=-N}^{N}\tilde{G}_{n(2\pi/X)}
\end{aligned}$$

It immediately follows that $\frac{1}{X}\sum_{n=-\infty}^{\infty}\tilde{G}_{n(2\pi/X)} \in \mathcal{U}$ and is equal to $\tilde{H}$. Thus part (i) part and (ii) are established.

(iii) Let f_N as above. For any function $g(x)$, with $g(x)/(1+|x|)^M$ of bounded variation on $(-\infty,\infty)$ for some $M>0$, and any $\psi(x)\in S$

$$|g(x)\psi(x)| < c/(1+|x|)^2$$

for some $c>0$. Hence,

$$\int_{-\infty}^{\infty} g(x) f_N(x)\psi(x)dx$$
$$= \lim_{K\to\infty}\left\{\int_{-(K+1/2)X}^{(K+1/2)X} f_N(x)g(x)\psi(x)dx\right\}$$
$$= \lim_{K\to\infty}\int_{-X/2}^{X/2} f_N(x)$$
$$\left\{\sum_{k=-K}^{K} g(x+kX)\psi(x+kX)\right\}dx$$

In addition, for any x,

$$|g(x+kX)\psi(x+kX)| < c/(1+|kX|)^2$$

for some $c>0$ and the series

$$\phi_K(x) = \sum_{k=-K}^{K} g(x+kX)\psi(x+kX)$$

is absolutely convergent. Hence, there exists a function, $\phi(x)$, such that $\phi_K(x)$ converges pointwise to $\phi(x)$ and there exists a constant, A, such that, for all $K>0$, $|\phi_K(x)|<A$, $\forall x\in[-X/2,X/2]$. Consequently, by Theorem 4.1 of (D.C.Champeney, 1987),

$$\lim_{K\to\infty}\int_{-X/2}^{X/2} f_N(x)\left\{\sum_{k=-K}^{K} g(x+kX)\psi(x+kX)\right\}dx$$
$$= \int_{-X/2}^{X/2} f_N(x)\phi(x)dx$$
$$= \int_{-X/2}^{X/2}\left(\frac{sin\left(\frac{\pi(2N+1)x}{2X}\right)}{x}\right)\left\{\frac{\phi(x)}{\left(\frac{sin\left(\frac{\pi x}{2X}\right)}{x}\right)}\right\}dx$$

Furthermore, $\phi(x)x/(sin(\pi x/(2X))$ is of bounded variation on $[-X/2,X/2]$. By Theorem 5.10 of (D.C.Champeney, 1987), $x=0$ is a Dirichlet point and

$$\lim_{N\to\infty}\int_{-X/2}^{X/2}\left(\frac{sin\left(\frac{\pi(2N+1)x}{2X}\right)}{x}\right)$$
$$\left\{\frac{\phi_k(x)x}{sin\left(\frac{\pi x}{2X}\right)}\right\}dx = X(\phi_k(0^+)+\phi_k(0^-))/2$$

Since, for $|x|<X/2$,

$$|g(kX+x)\psi(kX+x)| < c/(1+|kX|)^2$$

for some $c>0$

$$\phi(0^+) = \sum_{k=-\infty}^{\infty} g(kX^+)\psi(kX^+)$$

and

$$\phi(0^-) = \sum_{k=-\infty}^{\infty} g(kX^-)\psi(kX^-)$$

and it follows that

$$\lim_{N\to\infty}\int_{-\infty}^{\infty} f_N(x)g(x)\psi(x)dx$$
$$= \frac{1}{2}X\sum_{k=-\infty}^{\infty}((g(kX^+)\psi(kX^+))$$
$$+(g(kX^-)\psi(kX^-)))$$
$$= \frac{1}{2}X\sum_{k=-\infty}^{\infty}(g(kX^+)+g(kX^-)\psi(kX^-))\psi(kX)$$

Let $\tilde{f}_N \in \mathcal{D}_S$ be the regular functional defined by $f_N(x)$ then $\tilde{f}_N$ is a multiplier on $\mathcal{D}_S$. For the regular functional $\tilde{g}\in\mathcal{D}_S$ defined by $g(x)$

$$(\tilde{f}_N\tilde{g})[\psi(x)] = \tilde{g}[f_N(x)\psi(x)] = \int_{-\infty}^{\infty} g(x) f_N(x)\psi(x)dx$$

From the foregoing, it follows that

$$\lim_{N\to\infty}(\tilde{f}_N\tilde{g})[\psi(x)]$$
$$= X\sum_{k=-\infty}^{\infty}\frac{1}{2}(g(kX^-)+g(kX^+))\psi(kX)$$
$$= X\sum_{k=-\infty}^{\infty}\frac{1}{2}(g(kX^-)+g(kX^+))\tilde{\delta}_{kX}[\psi(x)]$$

Hence, $\frac{1}{X}\tilde{f}_N\tilde{g}$ converges to

$$\tilde{h} = \sum_{k=-\infty}^{\infty}\frac{1}{2}(g(kX^-)+g(kX^+))\tilde{\delta}_{kX}$$

in $\mathcal{D}_S$. Furthermore,

$$\tilde{H} = \mathcal{F}\{\tilde{h}\} = \sum_{k=-\infty}^{\infty}\frac{1}{2}(g(kX^-)+g(kX^+))\tilde{e}_{k(2\pi/X)} \in \mathcal{S}$$

and by Theorem 16.3 of (D.C.Champeney, 1987), $\tilde{H}$ is periodic with period $2\pi/X$ and Fourier coefficients $\left\{\frac{1}{2}(g(kX^-)+g(kX^+))\right\}$. However

$$\mathcal{F}\left\{\frac{1}{X}\tilde{f}_N\tilde{g}\right\} = \frac{1}{X}\mathcal{F}\{\tilde{f}_N\}*\mathcal{F}\{\tilde{g}\}$$
$$= \frac{1}{X}\left(\sum_{n=-N}^{N}\tilde{\delta}_{n(2\pi/X)}\right)*\tilde{G} = \frac{1}{N}\sum_{n=-N}^{N}\tilde{G}_{n(2\pi/X)}$$

It immediately follows that $\frac{1}{X}\sum_{n=-\infty}^{\infty}\tilde{G}_{n(2\pi/X)} \in \mathcal{D}_S$ and is equal to $\tilde{H}$. Thus part (iii) is established. ■