THE HEAT EQUATION AND THE FRENET FORMULAS FOR DIGITAL CURVES

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Abstract : In this note we shall discuss the heat equation and the eigenvalue problem for digital curves in the 3D Euclidean space. First, we shall introduce the derivative of a function along a digital curve by the weighted combination method. Then, we can define the Laplace of a function on a digital curve. The Frenet formulas for digital curves will also be discussed. Numerical simulations show our method will provide good estimations for the curvature and torsion.

1 INTRODUCTION

An ordered set of points $C = \{p_i \in R^3 \mid i = 1,2,\cdots,k\}$ is called a digital curve in the 3D Euclidean space R^3. The digital curves can be obtained by the discretization of regular curves or from digital images. Understanding the geometric and differential properties of digital curves is an important topic in CAD or CAGD. In particular, the curvature or the heat flow on a regular curve C in the 3D Euclidean space are important differential invariants in the theory of space curves and its applications to image processing and computer graphics. The curvature and torsion are determined by the differential of the tangent vectors and the binormal vectors of the curve C.

In this paper we shall discuss a differential theory for digital curves in the 3D Euclidean space. We shall discuss the derivative of a function along a digital curve by the weighted combination method. We shall use the centroid weights in our algorithms. These weights were first proposed in (Chen and Wu, 2004) to improve Taubin's method for the estimation of curvatures on a triangular mesh in the 3D Euclidean space. Then, we shall investigate the heat flow and the eigenvalue problem on digital curves. In section four, we shall discuss the moving frame of a digital curve and obtain the discrete Frenet formulas. This method fits perfectly with the proposal given in (Rosenfeld and Klette, 2002) about the field of digital geometry. Usually, the accurate estimation of curvatures at vertices of a digital curve plays as the first step for many applications such as simplification, smoothing, subdivision, visualization and image processing, etc. Our estimation is simple and very accurate as we shall illustrate them in the numerical simulations.

2 THE LOCAL THEORY FOR REGULAR CURVES

In this section we first recall some basic notions and results about the local theory of smooth regular curves in the 3D Euclidean space R^3. See (do Carmo, 1976) for details. Consider a smooth regular curve $c(s) = (x(s), y(s), z(s))$, $s \in [0,l]$ with arc length parameter s.

Given a function $f(s)$ on $c(s)$, we can define the Laplacian $\Delta f(s)$ of f by

$$\Delta f(s) = \frac{d^2}{ds^2} f(s). \qquad (2\text{-}1)$$

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The eigenvalue problem is given by

$$\Delta f = \lambda f \ . \tag{2-2}$$

The heat equation for the function $u(s,t)$ is given by

$$\frac{\partial}{\partial t}u = \Delta u \ . \tag{2-3}$$

Next we discuss the curvature and torsion of the regular curve $c(s) = (x(s), y(s), z(s))$ in R^3 with arc length parameter s. The tangent vector $c'(s) = (x'(s), y'(s), z'(s))$, denoted by $\vec{t}(s)$, is a unit vector since s is the arc length parameter. The number $\|\vec{t}\,'(s)\| = \kappa(s)$ is called the curvature of c at s. At points where $\kappa(s) \neq 0$, a unit vector $\vec{n}(s)$ in the direction $\vec{t}\,'(s)$ is well-defined by the equation $\vec{t}\,'(s) = \kappa(s)\vec{n}(s)$. The vector $\vec{n}(s)$ is perpendicular to $\vec{t}(s)$ and is called the normal vector of c at s. The plane determined by the unit tangent vector $\vec{t}(s)$ and normal vectors $\vec{n}(s)$ is called the osculating plane of c at s. At points where $\kappa(s) = 0$, the normal vector and hence the osculating plane are not defined. In what follows, we shall restrict ourselves to curves parametrized by arc length with $\kappa(s) \neq 0$ for all $s \in [0, l]$. The unit vector $\vec{b}(s) = \vec{t}(s) \times \vec{n}(s)$ is normal to the osculating plane and will be called the binormal vector of c at s. The number $\|\vec{b}\,'(s)\|$ measures the rate of change of the neighboring osculating planes of C at s. That is, $\|\vec{b}\,'(s)\|$ measures how rapidly the curves pulled away from the osculating plane of c at s in a neighborhood of s. The Frenet formulas are

$$\vec{t}\,'(s) = \kappa(s)\vec{n}(s) \tag{2-4}$$

$$\vec{n}\,'(s) = -\kappa(s)\vec{t}(s) - \tau(s)\vec{b}(s) \tag{2-5}$$

$$\vec{b}\,'(s) = \tau(s)\vec{n}(s) \tag{2-6}$$

These Frenet formulas form a system of ordinary differential equations (ODE's) for the vectors $\vec{t}(s)$, $\vec{n}(s)$ and $\vec{b}(s)$. We shall call the matrix

$$F(s) = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\tau(s) & 0 & -\kappa(s) \\ 0 & \tau(s) & 0 \end{pmatrix} \tag{2-7}$$

the Frenet matrix of the curve c at s.

3 A DISCRETE HEAT EQUATION FOR A DIGITAL CURVE

In this section we shall introduce a discrete heat equation for a digital curve in the 3D Euclidean space R^3. A digital curve C in the 3D Euclidean space is an ordered set of points $C = \{p_i \in R^3 : i = 1, 2, \cdots, k\}$. Consider a function f on the digital curve C. We can define the discrete derivative of the function f by:

$$\begin{aligned} \frac{d}{dx}f(p_i) \quad &= \omega_1 \frac{f(p_i) - f(p_{i-1})}{\|p_i - p_{i-1}\|} \\ &+ \omega_2 \frac{f(p_{i+1}) - f(p_i)}{\|p_{i+1} - p_i\|} \end{aligned} \tag{3.1}$$

when the point p_i is an interior point. When p_i is a boundary point i.e., $i = 0$ or k, we take the one-side derivative:

$$\frac{d}{dx}f(p_1) = \frac{f(p_2) - f(p_1)}{\|p_2 - p_1\|} \tag{3-2}$$

and

$$\frac{d}{dx}f(p_k) = \frac{f(p_k) - f(p_{k-1})}{\|p_k - p_{k-1}\|} \tag{3-3}$$

Indeed, when we know how to compute the derivatives of functions on a digital curve C, we can also compute their higher order derivatives. From the experience given in (Chen and Wu, 2004), (Chen and Wu, 2005) and (Wu, Chen and Chi 2005), we shall use the centroid weights for the weights ω_1 and ω_2. Namely, for the digital curve $C = \{p_i \in R^3 : i = 1, 2, \cdots, k\}$, we have at the point p_i

$$\begin{cases} \omega_1 = \dfrac{\dfrac{1}{\|p_i - p_{i-1}\|^2}}{\left(\dfrac{1}{\|p_i - p_{i-1}\|^2} + \dfrac{1}{\|p_{i+1} - p_i\|^2}\right)} \\ \\ \omega_2 = \dfrac{\dfrac{1}{\|p_{i+1} - p_i\|^2}}{\left(\dfrac{1}{\|p_i - p_{i-1}\|^2} + \dfrac{1}{\|p_{i+1} - p_i\|^2}\right)} \end{cases} \tag{3-4}$$

From this we can consider the discrete heat equation on C for a function $u(p_i, t)$:

$$\frac{\partial}{\partial t}u = \frac{\partial^2}{\partial x^2}u\,. \qquad (3\text{-}5)$$

If we consider the vector $v(t) = (u(p_i, t))^T$ in R^k, a direct computation of (3-4) will lead to a system of ODE's:

$$\frac{d}{dt}v(t) = A_k v(t) \qquad (3\text{-}6)$$

where $A_k = (a_{ij})$ is a $k \times k$ matrix with constant a_{ij}. The constant a_{ij} depends only on the points on C. From the theory of differential equations, the solution for $v(t) = (u(p_i, t))^T$ will have the form:

$$v(t) = e^{tA_k}v(0) \qquad (3\text{-}7)$$

with the initial value $v(0) = (u(p_i, 0))^T$. The matrix e^{tA_k} can be computed from the formula

$$e^{tA_k} = \sum_{n=0}^{\infty}\frac{(tA_k)^n}{n!} \qquad (3\text{-}8)$$

When the matrix A_k is symmetric, it is diagonalizable and one can find an orthogonal $k \times k$ matrix Q and a diagonal $k \times k$ matrix D such that $D = Q^T A_k Q$. Note that the column vectors of the orthogonal matrix Q are eigenvectors of A_k and the diagonal matrix $D = diag(\lambda_j)$ is given by the corresponding eigenvalues of A_k. In this case the solution $v(t)$ can be obtained from

$$v(t) = Q^T diag(e^{t\lambda_j})Qv(0) \qquad (3\text{-}9)$$

It can be shown easily that the proposed discrete heat density converges to the real heat density if a smooth curve is sampled finer and finer. This is also true for the curvature and torsion as one can see from the numerical simulations in section 5.

To illustrate our ideas, we consider the digital curve to be uniformly distributed and closed. Namely, the digital curve $C = \{p_i \in R^3 : i = 0,1,\cdots,k\}$ has constant distances $h = \|p_i - p_{i+1}\|$ for all $i = 0,1,\cdots,k-1$ and $p_0 = p_k$, An easy computation gives the $k \times k$ matrix $A_k = \frac{1}{4h^2}(a_{ij})$ with $a_{ii} = -2$, $a_{ij} = 1$ when $|i - j| = 2 \mod(k)$; otherwise, $a_{ij} = 0$.

In particular, the matrix A_k is symmetric and diagonalizable.

Therefore to study the discrete heat equation (3-5), we are led to the matrix eigenvalue problem

$$A_k v = \lambda v\,. \qquad (3\text{-}10)$$

To obtain the eigenvalues and their corresponding eigenvectors of A_k, we can transform the matrix A_k into a double stochastic matrix B_k by

$$B_k = 2h^2 A_k + I_k \qquad (3\text{-}11)$$

where I_k is the $k \times k$ identity matrix. This means that the double stochastic matrix $B_k = (b_{ij})$ has $b_{ij} = \frac{1}{2}$ when $|i - j| = 2 \mod(k)$; otherwise $b_{ij} = 0$.

When k is odd, we can permute the order of the coordinates by $1,3,\cdots,k,2,4,\cdots,k-1$ to obtain a new double stochastic matrix $C_k = (c_{ij})$ with has $c_{ij} = \frac{1}{2}$ when $|i - j| = 1 \mod(k)$; otherwise, $c_{ij} = 0$:

$$C = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \cdots & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & \cdots & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \cdots & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \cdots & 0 & \frac{1}{2} & 0 \end{bmatrix}$$

We have $C_k = P^T B_k QP$ for some permutation matrix P. We note that the graph associated with the double stochastic matrix C_k is a k-polygon (see Figure 1).

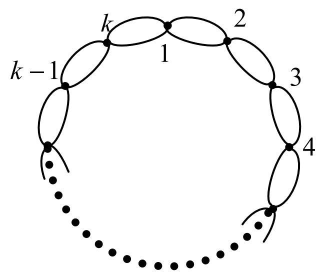

Figure 1: k-polygon.

When k is even, we can permute the order of the coordinates by $1,3,\cdots,k-1,2,4,\cdots,k$ and obtain a new double stochastic matrix D_k. Indeed, the matrix D_k decomposes into two blocks:

$$D_k = \begin{bmatrix} C_{\frac{k}{2}} & 0 \\ 0 & C_{\frac{k}{2}} \end{bmatrix} \quad (3\text{-}12)$$

where $C_{k/2}$ is as above. The graph associated with the double stochastic matrix D_k is two separated (k/2)-polygons (see Figure 2).

This gives that the eigenvalues and their corresponding eigenvectors of D_k can be obtained from those of the double stochastic matrix $C_{k/2}$. The double stochastic matrix C_n has the eigenvalues

$$\lambda_j = \cos(2j\pi/n), \quad j = 0,1,\ldots,n-1. \quad (3\text{-}13)$$

See (Bjorck and Golub, 1997). Every eigenvalue of the matrix C_n has multiplicity 2 except the eigenvalue 1, and if n is even also 1. Therefore, when k is odd, the number λ_j is also the eigenvalue of the double stochastic matrix B_k .In turn, the matrix A_k has the eigenvalues:

$$\lambda_j = \frac{1}{2h^2}(\cos(2j\pi/n)-1), \quad j = 0,1,\ldots,k-1. \quad (3\text{-}14)$$

Every eigenvalue of the matrix A_k has multiplicity 2 except the eigenvalue 0.

When k is even, the matrix D_k has the eigenvalues

$$\lambda_j = \cos(4j\pi/k), \quad j = 0,1,\ldots,k-1. \quad (3\text{-}15)$$

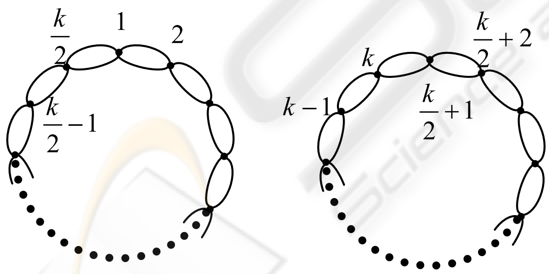

Figure 2: (k/2)-polygons.

Every eigenvalue of the matrix D_k has multiplicity 4 except the eigenvalue 1, and if $k/2$ is even also 1. When $k/2$ is odd, the eigenvalue 1 has multiplicity2. Hence the matrix A_k has the eigenvalues

$$\lambda_j = \frac{1}{2h^2}(\cos(4j\pi/k)-1), \; j = 0,1,\ldots,k-1. \quad (3\text{-}16)$$

4 DISCRETE FRENET FORMULAS FOR A DIGITAL CURVE

In this section we shall propose an algorithm to develop a discrete Frenet matrix for a digital curve. Recall that a digital curve C in the 3D Euclidean space is an ordered set of points $C = \{p_i \in R^3 : i = 1,2,\cdots,k\}$. To define the tangent vector $\vec{t}_i$ and the normal vector $\vec{n}_i$ and the binormal vector $\vec{b}_i$ of the digital curve C at the point p_i is the first step to develop a geometric theory for digital curves. To handle this, we need to formulate the concept of the derivative of a vector field defined on a digital curve C.

Consider a point p_i in the digital curve C. We can define the tangent vector $\vec{t}_i$ of C at the point p_i by

$$\vec{t}_i = \frac{(\omega_1 \frac{p_i - p_{i-1}}{\|p_i - p_{i-1}\|} + \omega_2 \frac{p_{i+1} - p_i}{\|p_{i+1} - p_i\|})}{\left\| \omega_1 \frac{p_i - p_{i-1}}{\|p_i - p_{i-1}\|} + \omega_2 \frac{p_{i+1} - p_i}{\|p_{i+1} - p_i\|} \right\|} \quad (4\text{-}1)$$

where ω_1 and ω_2 are the centroid weights given in (3-3). Now the normal vector $\vec{n}_i$ can be computed as follows: First we compute the derivative $\vec{t}_i^{'}$ of the tangent field $\vec{t}_i$ of C at the point p_i by

$$\vec{t}_i^{'} = \omega_1 \frac{\vec{t}_i - \vec{t}_{i-1}}{\|p_i - p_{i-1}\|} + \omega_2 \frac{\vec{t}_{i+1} - \vec{t}_i}{\|p_{i+1} - p_i\|}. \quad (4\text{-}2)$$

Note that the vector $\vec{t}_i^{'}$ may not be perpendicular to the tangent vector $\vec{t}_i$. We can define the curvature κ_i and the normal vector $\vec{n}_i$ of the digital curve C at p_i by

$$\begin{cases} \kappa_i = \|\vec{t}_i^{'} - (\vec{t}_i^{'} \cdot \vec{t}_i)\vec{t}_i\| \\ \vec{n}_i = \dfrac{(\vec{t}_i^{'} - (\vec{t}_i^{'} \cdot \vec{t}_i)\vec{t}_i)}{\|\vec{t}_i^{'} - (\vec{t}_i^{'} \cdot \vec{t}_i)\vec{t}_i\|} \end{cases}. \quad (4\text{-}3)$$

As usual, the binormal vector $\vec{b}_i$ of the digital curve C at p_i can be defined by $\vec{b}_i = \vec{t}_i \times \vec{n}_i$. Next we consider the torsion τ_i of the digital curve C at p_i via the derivative of the binormal vector field $\vec{b}_i$.

We have

$$\vec{b}_i' = \omega_1 \frac{\vec{b}_i - \vec{b}_{i-1}}{\|p_i - p_{i-1}\|} + \omega_2 \frac{\vec{b}_{i+1} - \vec{b}_i}{\|p_{i+1} - p_i\|} \quad (4\text{-}4)$$

and the torsion τ_i can be defined by $\tau_i = \vec{b}_i' \cdot \vec{n}_i$. The discrete version of the Frenet formulas will then have the form：

$$\vec{t}_i' = a_{11}\vec{t}_i + \kappa_i \vec{n}_i \quad (4\text{-}5)$$

$$\vec{n}_i' = a_{21}\vec{t}_i + a_{22}\vec{n}_i + a_{23}\vec{b}_i \quad (4\text{-}6)$$

$$\vec{b}_i' = a_{31}\vec{t}_i + \tau_i \vec{n}_i + a_{33}\vec{b}_i \quad (4\text{-}7)$$

where the coefficients a_{ij} may not be zero. This is due to the discrete effect of the digital curve C. We define the discrete Frenet matrix of the digital curve C at p_i to be the 3×3 matrix:

$$F = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad (4\text{-}8)$$

where a_{ij} is given by equations (4-5), (4-6) and (4-7).

5 NUMERICAL SIMULATIONS

In this section, we will find the Frenet matrices of the closed curves (without boundary points) and the open curves (with two boundary points). For closed curves, we choose the ellipses and C^2 Bezier curves. For open curves, we choose the helix

$$c(t) = (a\cos\frac{s}{c}, a\sin\frac{s}{c}, \frac{b}{c}) \quad (5\text{-}1)$$

with $a,b>0$ and $c=\sqrt{a^2+b^2}$. We shall compare the error between the exact Frenet matrix and our estimated discrete Frenet matrix by

$$Error = \frac{\| RF - F \|}{\| RF \|} \quad (5\text{-}2)$$

where RF is the exact Frenet matrix of the given regular curve and $\|\bullet\|$ is the norm of matrix. We will digitize these curves by two different kinds of partitions -- uniform and non-uniform partitions. In figures 3 to 8, the x-axis presents the number of points of digital curves and the y-axis gives the average of errors. We test 1,000 different random curves in each partition for different size of points and compute their average.

In figures 3 and 4, we show the numerical results of closed curves and helix by uniform partitions. From these results, the discrete Frenet matrix approximates to the exact Frenet matrix very quickly. In figures 5 and 8, we test the helix with uniform or non-uniform partitions at the interior points and the boundary points. These numerical simulations show that our discrete method is very stable.

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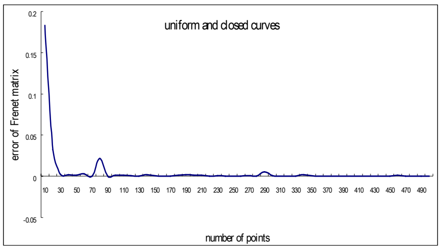

Figure 3: Closed curves with uniform partitions.

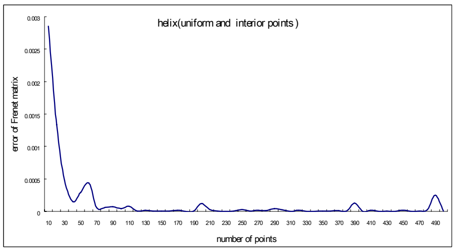

Figure 4: Interior points with uniform partitions.

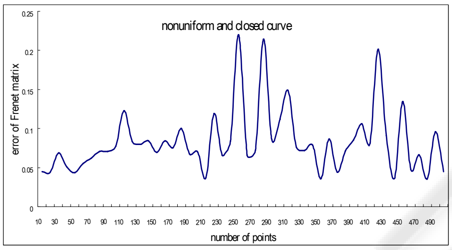

Figure 5: Closed curves with non-uniform partitions.

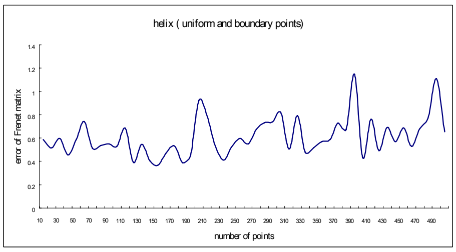

Figure 6: Boundary points with uniform partitions.

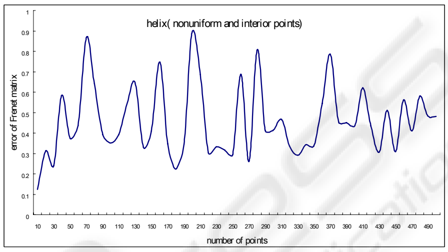

Figure 7: Interior points with non-uniform partitions.

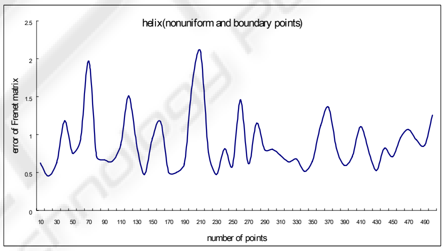

Figure 8: Boundary points with non-uniform partitions.