EXPONENTIAL OBSERVER FOR A CLASS OF NONLINEAR DISTRIBUTED PARAMETER SYSTEMS WITH APPLICATION TO A NONISOTHERMAL TUBULAR REACTOR

Nadia Barje, Mohammed Achhab
Laboratoire d'Ingénierie Mathematique (LINMA), Université Chouaib Doukkali, El Jadida, Morroco

Vincent Wertz *
CESAME, Universit de Louvain, Louvain-la-Neuve, Belgium

Keywords: Nonlinear infinite-dimensional systems, state observer, nonlinear tubular reactor.

Abstract: This paper present sufficient conditions to construct an exponential state estimator for a class of infinite dimensional non-linear systems driven in a real Hilbert state description. The theory is applied to a nonisothermal plug flow tubular reactor, governed by hyperbolic first order partial differential equations. For this application performance issues of the exponential state estimator design are illustrated in a simulation study.

1 INTRODUCTION

State estimators for dynamical systems have been the focus of an intensive work in the last decades. The classical theory of the Luenberger observers has been successfully extended from finite dimensional linear systems to a large class of infinite dimensional linear systems by many authors, since the pioneering paper by (Gressang and Lamont, 1975). Later, the theory has been generalized to a class of single input distributed bilinear systems in (Gauthier et al., 1995). The paper of (Bounit and Hammouri, 1997) consider a class of distributed bilinear systems witch are observable for "small inputs" and gives a strong exponential observer. Recently, for nonlinear models of non-isothermal tubular reactors considered in (Laabissi et al., 2001), the paper of (Orlov and Dochain, 2002) presented a reduced-order observer of the concentration, assuming that the temperature is the only available measurement.

The primary objective of this paper is to address the problem of the design of exponential Luenberger-like observers for a class of infinite dimensional nonlinear systems described by the following equation

$$\begin{cases} \dot{x}(t) = Ax(t) + N(x(t)), x(0) \in D(A) \cap D \\ y(t) = Cx(t) \end{cases} \quad (1)$$

Here, A is the infinitesimal generator of a C_0-semigroup on a real Hilbert space H with inner product $< .,. >$ and norm $\| . \|$, $D(A)$ is the domain of A, N is a nonlinear operator from a closed subset D of H into H, $y(t) \in Y$ is the known output function associated to the unknown initial condition $x(0)$, Y is another real Hilbert space and C is a bounded linear operator from H into the Hilbert space Y. Under the assumption that N is locally Lipschitz continuous, it is shown in ((Pazy, 1983), pp. 185-186) that equation (1) has a unique mild solution on some interval $[0, t_{max})$, $t_{max} \in (0, +\infty]$ given by

$$\begin{cases} x(t) = S(t)x(0) + \int_0^t S(t-s)N(x(s))ds, \\ 0 \le t < t_{max} \end{cases} \quad (2)$$

where $(S(t))_{t\ge 0}$ denotes the C_0-semigroup generated by A. To ensure that the problem is well posed, we shall assume throughout the paper as in (Laabissi et al., 2001) that we have: $t_{max} = +\infty$. An observer design is presented for which a result about the exponential convergence of the estimation error is stated under verifiable conditions.

Our second objective is to apply the previous developed result to the nonlinear model of a chemical plug flow reactor. This model is showed to be described

*This work was finalized when this author was on sabbatical leave at the ARC Centre of Excellence for Complex Dynamic Systems and Control, The University of Newcastle, Australia, whose support is gratefully acknowledged.

Barje N., Achhab M. and Wertz V. (2008).
EXPONENTIAL OBSERVER FOR A CLASS OF NONLINEAR DISTRIBUTED PARAMETER SYSTEMS WITH APPLICATION TO A NONISOTHERMAL TUBULAR REACTOR.
In *Proceedings of the Fifth International Conference on Informatics in Control, Automation and Robotics - ICSO*, pages 20-27
DOI: 10.5220/0001484900200027

by (1). Trajectory analysis of such a model of chemical plug flow reactors has been done extensively in (Achhab et al., 1999) and (Laabissi et al., 2001). For this application, we also introduce a second observer in the case when only one of the two states, namely the temperature, is measured and show the exponential convergence of both estimation errors. A third observer is then introduced to improve the convergence rate of the previous one. Simulations results are then presented in order to highlight the performance issues of the proposed observers.
The paper is organized as follows. In section 2, we consider a general observer design for system (1). Then, we state sufficient conditions under which the related estimation error converges exponentially to zero. The approach developed in the general setting is applied to a chemical plug flow reactor model in section 3. In section 4, simulation results are given in order to illustrate some performance issues of this application. Finally, the paper closes with some remarks and conclusions in section 5. The background of the approach is to be found in (Curtain and Zwart, 1995) and (Cazenave and Haraux, 1998).

2 OBSERVER DESIGN

We state in this section sufficient conditions under which we will be able to show that the estimation error of the Luenberger-like observer converges exponentially to zero.
Let us assume that the following.

A.1. The linear operator A satisfies for all $x \in D(A)$, and $t \geq 0$, $< Ax(t), x(t) > \leq 0$.

A.2. The nonlinear operator N is a k_N-Lipschitz operator on its domain D, where k_N is a positive constant; i.e. for all $x, y \in D$, $\| N(x) - N(y) \| \leq k_N \| x - y \|$.

A.3. The pair (A,C) is approximately observable linear system (i.e. $\forall e \in H$, $\{CS(t)e = 0, \ \forall t \geq 0$, implies $e = 0\}$), exponentially stable.

A.4. The semigroup $S(.)$ satisfies for all $e \in H$:

$$< S^*(.)C^*CS(.)e, e > \leq \| S(.) \|^2 < C^*Ce, e >,$$

Comment 2.1. *The hypothesis A.3 implies that the linear system*

$$\begin{cases} \dot{x}(t) = Ax(t), x(0) \in D(A) \\ y(t) = Cx(t) \end{cases}$$

is approximately observable on $[0, +\infty)$ *and that the observability gramian* $L_C := C^*C$, *where* $Ce := CS(.)e$ *for all* $e \in H$, *and* C^* *is the adjoint of the linear operator* C, *is bounded positive definite (see (Curtain and Zwart, 1995), p.160), and thus has an algebraic bounded inverse with domain equal to range* L_C.

Consider now the following candidate observer

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + N(\hat{x}(t)) - GC^*(C\hat{x}(t) - y(t)) \\ \hat{x}(0) \in D(A) \cap D \end{cases} \quad (3)$$

where G is a linear bounded operator and $y(t)$ is the known output function of the system (1). One can show that system (3) admits a unique solution $\hat{x}(t)$ which is well defined for any initial condition $\hat{x}(0) \in D(A) \cap D$ and for all $t \in [0, t_{max})$, with t_{max} assumed to be equal $+\infty$.
Setting $e(t) = x(t) - \hat{x}(t)$, the reconstruction error $e(t)$ obeys the following equation:

$$\dot{e}(t) = Ae(t) + N(x(t)) - N(\hat{x}(t)) - GC^*Ce(t) \quad (4)$$

and one obtains the following theorem:

Theorem 2.1. *Let assumptions A.1-A.4 be satisfied. If there exists a bounded linear operator* G *and a positive real number* g *such that* $g > k_N$ *and for* $e \in H$, $e \neq 0$,

$$< GC^*Ce, e > \geq < g \| L_C^{-1} \| \| S(.) \|^2 C^*Ce, e >$$

then, system (3) *is an exponential observer for system* (1). *More precisely, the reconstruction error satisfies* $\| e(t) \|^2 \leq \| e(0) \|^2 e^{-\eta t}$ *where* $\eta = 2(g - k_N)$.

Proof 2.1. *The computation of the derivative of the functional*

$$V_e(t) = \frac{1}{2} \| e(t) \|^2$$

along the trajectories of (4) *yields,*

$$\begin{aligned} \dot{V}_e(t) &= < \dot{e}(t), e(t) > \\ &= < Ae(t), e(t) > + < N(x(t)) - N(\hat{x}(t)), e(t) > \\ &\quad - < GC^*Ce(t), e(t) > \end{aligned}$$

and in addition,

$$\begin{aligned} < GC^*Ce, e > &\geq g \| L_C^{-1} \| \| S(.) \|^2 < C^*Ce, e > \\ &\geq g \| L_C^{-1} \| < L_C e, e > \\ &\geq g < e, e > \end{aligned}$$

indeed, the operator L_C *is self-adjoint and nonnegative (i.e,* $< L_C e, e > \geq 0$ *for all* $e \in H$*), then* L_C *has a unique square root* $L_C^{\frac{1}{2}}$ *self-adjoint, so that* $L_C^{\frac{1}{2}} L_C^{\frac{1}{2}} e = L_C e$ *for all* $e \in H$ *(see (Curtain and Zwart, 1995), p.606), the operator* L_C^{-1} *is also self-adjoint and nonnegative, par consequent has a unique square root* $(L_C^{-1})^{\frac{1}{2}} = (L_C^{\frac{1}{2}})^{-1}$ *(see (Curtain and Zwart, 1995), pp. 603-610).*
and in addition, for all $e \in H$,

$$< L_C^{-1} e, e > \quad \leq \| L_C^{-1} \| < e, e >$$

thus,

$$\begin{aligned}\| L_C^{-1} \| < L_C e, e > &= \| L_C^{-1} \| < L_C^{\frac{1}{2}} L_C^{\frac{1}{2}} e, e > \\ &= \| L_C^{-1} \| < L_C^{\frac{1}{2}} e, L_C^{\frac{1}{2}} e > \\ &\geq < L_C^{-1} L_C^{\frac{1}{2}} e, L_C^{\frac{1}{2}} e > \\ &\geq < (L_C^{\frac{1}{2}})^{-1} L_C^{\frac{1}{2}} e, (L_C^{\frac{1}{2}})^{-1} L_C^{\frac{1}{2}} e > \\ &= < e, e >\end{aligned}$$

Hence,

$$\begin{aligned}\dot{V}_e(t) &\leq \| N(x(t)) - N(\hat{x}(t)) \| \| e(t) \| - g \| e(t) \|^2 \\ &\leq (k_N - g) \| e(t) \|^2 = -\eta V_e(t)\end{aligned}$$

Now, using Gronwall's Lemma (see (Curtain and Zwart, 1995), p. 639), we get

$$V_e(t) \leq V_e(0) e^{-\eta t}$$

Consequently, one may deduce

$$\| e(t) \|^2 \leq \| e(0) \|^2 \, e^{-\eta t}$$

This shows the exponential convergence of the estimation error and the proof of Theorem (2.1) is thus complete.

2.1 Application to a Nonisothermal Plug-Flow Reactor

The theory developed in the general setting is applied to a chemical non-isothermal tubular reactor with the following chemical reaction:

$$A \rightarrow B$$

The kinetics of the above reaction is characterized by first-order kinetics with respect to the reactant concentration $C(mol/l)$ and by an Arrhenius-type dependence with respect to the temperature $T(K)$, and the dynamics of the process are described by the following two energy and mass balance PDEs (see (Laabissi et al., 2001)):

$$\frac{\partial T}{\partial \tau} = -\upsilon \frac{\partial T}{\partial \zeta} - \frac{4h}{\rho C_p d}(T - T_c) - \frac{\Delta H}{\rho C_p} k_0 C e^{\frac{-E}{RT}}, \quad (5)$$

$$\frac{\partial C}{\partial \tau} = -\upsilon \frac{\partial C}{\partial \zeta} - k_0 C e^{\frac{-E}{RT}}, \quad (6)$$

where the boundary conditions are given, for $\tau \geq 0$, by:

$$T(0,\tau) = T_{in}, \; C(0,\tau) = C_{in} \quad (7)$$

and the initial conditions are assumed to be given, for $0 \leq \zeta \leq L$, by:

$$T(\zeta,0) = T_0(\zeta), \; C(\zeta,0) = C_0(\zeta) \quad (8)$$

In the equations above, the following parameters υ, ΔH, ρ, C_p, k_0, E, R, h, d, T_c hold for the superficial fluid velocity, the heat of reaction, the density, the specific heat, the kinetic constant, the activation energy, the ideal gas constant, the wall heat transfer coefficient, the reactor diameter, the coolant temperature. T_{in} and C_{in} are respectively the inlet temperature and the inlet reactant concentration which will be assumed to be two positive constants. τ, ζ and L denote the time and space independent variables, and the length of the reactor, respectively. Finally T_0 and C_0 denote the initial temperature and reactant concentration profiles.

The dynamics will be described by means of an infinite-dimensional system derived from an equivalent nonlinear PDE dimensionless model. Such an approach is standard in tubular reactor analysis (see (Laabissi et al., 2001)) and is briefly developed here. Let here after $H = L^2[0,L] \times L^2[0,L]$, endowed by the inner product

$$< (x_1,x_2),(y_1,y_2) > = < x_1,y_1 >_{L^2} + < x_2,y_2 >_{L^2}$$

and the induced norm

$$\| (x_1,x_2) \| = (\| x_1 \|_{L^2}^2 + \| x_2 \|_{L^2}^2)^{\frac{1}{2}}$$

for all (x_1,x_2) and (y_1,y_2) in H. We will denote here after $<,>_{L^2}$ by $<,>$.

Consider the following dimensionless state variables:

$$x_1 = \frac{T - T_{in}}{T_{in}}, \quad x_c = \frac{T_c - T_{in}}{T_{in}}, \quad x_2 = \frac{C_{in} - C}{C_{in}}, \quad r(x_1) = e^{\frac{\mu x_1}{1+x_1}}$$

Let us consider also dimensionless time t and space z variables:

$$t = \frac{\tau\upsilon}{L}, \; z = \frac{\zeta}{L}.$$

We shall assume in the rest of the paper that the coolant temperature T_c is equal to the inlet temperature T_{in} (i.e $x_c \equiv 0$), since x_c will be eliminated in the equation of the reconstruction error between the plan state and the observer state.

Then we obtain the following equivalent representation of the model (5)-(8):

$$\frac{\partial x_1}{\partial t} = -\frac{\partial x_1}{\partial z} - \beta x_1 + \alpha\delta(1 - x_2) r(x_1) \quad (9)$$

$$\frac{\partial x_2}{\partial t} = -\frac{\partial x_2}{\partial z} + \alpha(1 - x_2) r(x_1) \quad (10)$$

with the boundary conditions:

$$x_1(z=0,t) = 0, \quad x_2(z=0,t) = 0 \quad (11)$$

and initial conditions

$$x_1(z,0) = x_1^0, \quad x_2(z,0) = x_2^0 \tag{12}$$

and the parameters α, β, δ and μ are related to the original parameters as follows:

$$\mu = \frac{E}{RT_{in}}, \quad \alpha = \frac{k_0 L}{\upsilon} exp(-\mu)$$
$$\beta = \frac{4hL}{\rho C_p d\upsilon}, \quad \delta = -\frac{\Delta H}{\rho C_p}\frac{C_{in}}{T_{in}}.$$

From a physical point of view it is expected that for all $z \in [0,1]$, and for all $t \geq 0$ (see (Aksikas et al., 2007)),

$$0 \leq T(z,t) \leq T_{max} \text{ and } 0 \leq C(z,t) \leq C_{in}$$

or equivalently

$$-1 \leq x_1(z,t) \leq \frac{T_{max} - T_{in}}{T_{in}} \text{ and } 0 \leq x_2(z,t) \leq 1,$$

where T_{max} could possibly be equal to $+\infty$.
This is also true for the model, as shown by (Laabissi et al., 2001).
The equivalent state space description of the model (9)-(12) is given by the following nonlinear abstract differential equation on the Hilbert space $H = L^2[0,1] \times L^2[0,1]$:

$$\begin{cases} \dot{x}(t) = Ax(t) + N(x(t)) \\ x(0) = x_0 \in D(A) \cap D \end{cases} \tag{13}$$

where, A is the linear operator defined on its domain

$$D(A) := \{x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \in H : x \text{ absolutely continuous, } \frac{dx}{dz} \in H \text{ and } x_i(0) = 0, i = 1,2\}$$

by,

$$Ax := \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{d.}{dz} - \beta. & 0 \\ 0 & -\frac{d.}{dz} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The linear operator A satisfies,

$$< Ax, x > \leq -\frac{1}{2}x^2(1), \text{ for all } x \in D(A)$$

which satisfies the hypothesis A.1.
A is the generator of a C_0-semigroup exponentially stable

$$S(t) = \begin{pmatrix} S_1(t) & 0 \\ 0 & S_2(t) \end{pmatrix}$$

satisfying (see (Winkin et al., 2000)), for all $(x_1, x_2) \in H$,

$$(S_1(t)x_1)(z) = \begin{cases} e^{-\beta t}x_1(z-t) & \text{if } z \geq t, \\ 0 & \text{if } z < t, \end{cases}$$

$$(S_2(t)x_2)(z) = \begin{cases} x_2(z-t) & \text{if } z \geq t, \\ 0 & \text{if } z < t, \end{cases}$$

Moreover, (see (Winkin et al., 2000))

$$\| S(t) \| \leq 1 \text{ for all } t \geq 0$$

The nonlinear operator N is defined on

$$D := \{x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \in H : -1 \leq x_1(z) \text{ and } 0 \leq x_2(z) \leq 1, \text{ for almost all } z \in [0,1]\} \text{ by,}$$

$N(x) = (N_1(x), N_2(x))^T$ where for all $x = (x_1, x_2)^T \in D$,

$$N_1(x) = \alpha\delta(1 - x_2)r(x_1)$$
$$N_2(x) = \alpha(1 - x_2)r(x_1)$$

It is proved in (Aksikas et al., 2007) that the function $m(s) := exp(\frac{-k}{s})$ where $k = \frac{E}{R}$, is a Lipschitz continuous function on $[0, T_{max}]$ with a Lipschitz constant l_s given by

$$l_s = \begin{cases} 4\frac{R}{Ee^2} & \text{if } E \leq 2RT_{max} \\ \frac{E}{RT_{max}^2} exp(-\frac{E}{RT_{max}}) & \text{if } E > 2RT_{max} \end{cases}$$

It follows that the constant $k_r := e^{\mu} T_{in} l_s$ is a Lipschitz constant for the function $r(s) := exp(\frac{\mu s}{1+s})$.
We prove that for all

$$x := (x_1, x_2)^T \text{ and } y := (y_1, y_2)^T \in D,$$
$$\| N_2(x) - N_2(y) \| \leq \alpha exp(\mu) \| x_2 - y_2 \| + \alpha k_r \| x_1 - y_1 \|,$$

Observe that $N_1 = \delta N_2$, thus we take $k_N := \alpha(exp(\mu) + k_r)(1 + |\delta|)$ as a Lipschitz constant of N, the hypothesis A. 2 is thus satisfied.
It is proved in (Laabissi et al., 2001) that the system Eq. 13 has a unique mild solution $x(t, x(t = 0))$ on $[0, +\infty[$, for all $x_0 \in D$ and that the state remains in D.
Hereafter we consider measurements at the reactor output. In this case, the output function $y(.)$ is defined as follows: we consider a (very small) finite interval at the reactor output $[1 - w, 1]$ such that:

$$y(t) = (Cx)(t) := \int_0^1 \chi_{[1-w,1]}(a) I_{2\times 2} x(a,t)da, \ \forall t \in \mathbb{R}^+$$

where

$$\chi_{[1-w,1]}(a) = \begin{cases} 1, & \text{if } a \in [1 - w, 1] \\ 0, & \text{elsewhere.} \end{cases}$$

and $I_{2\times 2}$ is either the 2×2 identity matrix operator when both components x_1 and x_2 are measured, or a unit row vector if only one of them is measured. In the first case (i.e. two measurements), it is proved in (Winkin et al., 2000), that the pair (C,A) is approximately observable if both x_1 and x_2 are measured at the reactor output, thus hypothesis A.3 is satisfied and so the observability gramian $L_C := C^*C$, where $Cx := CS(.)x$ for all $x \in H$ is positive definite and has an algebraic inverse L_C^{-1} with domain equal to *range* L_C, satisfying for $x_d(z,t) = I_d(z,t)$, where $I_d(z,t) = 1$ for all $(z,t) \in [0,1] \times \mathbb{R}^+$:

$$\begin{aligned} < L_C x_d, x_d > &= < CS(.)x_d, CS(.)x_d > \\ &\geq w^2 e^{-2\beta} \parallel x_d(z) \parallel^2 \end{aligned}$$

on have

$$\parallel L_C \parallel^2 \geq w^2 e^{-2\beta} \parallel L_C \parallel$$

and

$$< L_C x, L_C x > \geq w^2 e^{-2\beta} < L_C x, x >$$

Observe that L_C is self-adjoint and for all $y \in$ *range* L_C,

$$\begin{aligned} < L_C^{-1} y, y > &= < L_C^{-1} y, L_C L_C^{-1} y > \\ &\leq \frac{1}{w^2 e^{-2\beta}} < L_C L_C^{-1} y, L_C L_C^{-1} y > \\ &\leq \frac{1}{w^2 e^{-2\beta}} < y, y > \end{aligned}$$

this implies,

$$\parallel L_C^{-1} \parallel \leq \frac{1}{w^2 e^{-2\beta}}$$

Denote $w_0 = w^2 e^{-2\beta}$

A candidate Luenberger-observer for system (9)-(12) when the state variables are measured is

$$\begin{aligned} \frac{\partial \hat{x}_1}{\partial t} = \quad & -\frac{\partial \hat{x}_1}{\partial z} - \beta \hat{x}_1 + \alpha\delta(1-\hat{x}_2) r(\hat{x}_1) + \\ & \frac{g}{w_0} C_1^* C_1 e_1 \end{aligned} \tag{14}$$

$$\frac{\partial \hat{x}_2}{\partial t} = \quad -\frac{\partial \hat{x}_2}{\partial z} + \alpha(1-\hat{x}_2) r(\hat{x}_1) + \frac{g}{w_0} C_2^* C_2 e_2 \tag{15}$$

with the boundary conditions:

$$\hat{x}_1(z=0,t) = 0, \quad \hat{x}_2(z=0,t) = 0 \tag{16}$$

and initial conditions

$$\hat{x}_1(z,t=0) = \hat{x}_1^0, \quad \hat{x}_2(z,t=0) = \hat{x}_2^0 \tag{17}$$

with $C = \begin{pmatrix} C_1 & C_2 \end{pmatrix}$ and $e_i(z,t) = x_i(z,t) - \hat{x}_i(z,t)$ for $i = 1,2$, for all $(z,t) \in [0,1] \times \mathbb{R}^+$.

The observer state remains in the set D, the main steps of the proof go along the line of the one given in (Laabissi et al., 2001).

Observe that the model (14)-(17) is in the form of the nonlinear abstract differential equation (3), with the linear operator G chosen as follows: $G = \frac{g}{w_0} I$ where I is the identity operator and g is a positive real number.

For the bounded operator C given above, the C_0-semigroup $(S(t))_{t \geq 0}$ satisfies for all $e = (e_1, e_2)^T \in H$:

$$< S(.)C^*CS(.)e, e > \quad \leq \parallel S(.) \parallel^2 < C^*Ce, e >$$

which satisfies the hypothesis A.4.

Denote $e(t)$ the reconstruction error between the plant state and the observer state. A direct application of Theorem 2.1 yields the following result.

Corollaire 2.1. *Take g such that $g > k_N$ holds. Then, system* (14)-(17) *is an exponential observer for system* (9)-(12)*. More precisely, the reconstruction error $e(t)$ has the property that $\parallel e(t) \parallel^2 \leq \parallel e(0) \parallel^2 e^{-\eta t}$, where $\eta = 2(g - k_N)$.*

It is also interesting to examine the case where only the temperature $x_1(z,t)$ is measured at the output of the reactor. So, the observation operator $C = \begin{pmatrix} C_1 & C_2 \end{pmatrix}$ is given by

$$\begin{aligned} (C_1 e_1)(t) &= \int_0^1 \chi_{[1-w,1]}(a) e_1(a,t) da, \ \forall t \in \mathbb{R}^+ \\ C_2 &= 0 \end{aligned}$$

Recall that the linear operator A is diagonal. The operators A_1 and A_2 satisfy;

$$< A_i x_i, x_i > \leq -\frac{1}{2} x_i^2(1), \text{ for all } x_i \in D(A_i) \text{ and } i = 1,2$$

on their common domain:

$$\begin{aligned} D(A_{i=1,2}) &= \{x \in L^2[0,1] : x \text{ absolutely continuous}, \\ & \frac{dx}{dz} \in L^2[0,1] \text{ and } x(0) = 0\}. \end{aligned}$$

In the same manner as in (see (Winkin et al., 2000)), we prove that (C_1, A_1) is approximately observable.

In this case, a full-order observer for the dimensionless model (9)-(12) can be constructed as follows:

$$\begin{aligned} \frac{\partial \hat{x}_1}{\partial t} &= -\frac{\partial \hat{x}_1}{\partial z} - \beta \hat{x}_1 + \alpha\delta \sup_{x_2 \in D_2}(1 - x_2) r(\hat{x}_1) + \\ & \frac{g}{w_0} C_1^* C_1 e_1 \end{aligned} \tag{18}$$

$$\frac{\partial \hat{x}_2}{\partial t} = -\frac{\partial \hat{x}_2}{\partial z} + \alpha(1-\hat{x}_2) \sup_{x_1 \in D_1} r(x_1) \tag{19}$$

with the boundary conditions:

$$\hat{x}_1(z=0,t) = 0, \ \hat{x}_2(z=0,t) = 0 \tag{20}$$

and initial conditions

$$\hat{x}_1(z,t=0) = \hat{x}_1^0, \ \hat{x}_2(z,t=0) = \hat{x}_2^0 \tag{21}$$

Note that in this observer, the nonlinear term is not exactly taken as in the "true system", for technical reasons in the convergence proof.

It can be shown as in (Laabissi et al., 2001) that the observer states remains in the set D.

We state the following result:

Theorem 2.2. *Take g such that $g > k_r \alpha\delta$ holds. Then, system* (18)-(21) *is an exponential observer for the non-isothermal plug flow reactor model* (9)-(12)*. More precisely the reconstruction errors have the properties that $\parallel e_1(t) \parallel^2 \leq \parallel e_1(0) \parallel^2 e^{-\nu_1 t}$ and $\parallel e_2(t) \parallel^2 \leq \parallel e_2(0) \parallel^2 e^{-\nu_2 t}$, where $\nu_1 := 2(g - \alpha\delta k_r)$ and $\nu_2 = 2\alpha exp(\mu)$.*

The proof is similar to that given in Theorem 2.1.
The concentration error converges to zero with convergence rate ν_2 depending only on the internal dynamics of the process. It will be interesting to look for a "closed loop" observer design that will make ν_2 as large as desired. In this case however the full state will need to be observed. The following is given to improve the convergence rate of the concentration error.

To have an a priori given convergence rate of the concentration error $\nu_2^* \geq \nu_2$, one can use the following full order observer:

$$\frac{\partial \hat{x}_1}{\partial t} = -\frac{\partial \hat{x}_1}{\partial z} - \beta \hat{x}_1 + \alpha\delta \sup_{x_2 \in D_1}(1-x_2) r(\hat{x}_1) + \frac{m_1}{w_0} C_1^* C_1 e_1 \quad (22)$$

$$\frac{\partial \hat{x}_2}{\partial t} = -\frac{\partial \hat{x}_2}{\partial z} + \alpha(1-\hat{x}_2) \sup_{x_1 \in D_2} r(x_1) + \frac{m_2}{w_0} C_2^* C_2 e_2 \quad (23)$$

with the boundary conditions:

$$\hat{x}_1(z=0,t) = 0,\ \hat{x}_2(z=0,t) = 0 \quad (24)$$

and initial conditions

$$\hat{x}_1(z,t=0) = \hat{x}_1^0,\ \hat{x}_2(z,t=0) = \hat{x}_2^0 \quad (25)$$

where, for $i = 1,2$

$$(C_i e_i)(t) := \int_0^1 \chi_{[1-w,1]}(a) e_i(a,t) da,\ \forall t \in \mathbb{R}^+$$

where m_1 is a positive real number, $m_2 = \frac{\nu_2^* - \nu_2}{2}$. then we have the following result:

Theorem 2.3. *Consider the full-order observer* (22)-(25) *for the uncontrolled system* (9)-(12) *where* $m_1 > \alpha\delta k_r$. *Then the temperature error* $e_1(t)$ *satisfies* $\| e_1(t) \|^2 \leq \| e_1(0) \|^2 e^{-\nu_1 t}$, *where* $\nu_1 := 2(m_1 - \alpha\delta k_r)$, *and the concentration error* $e_2(t)$ *satisfies* $\| e_2(t) \|^2 \leq \| e_2(0) \|^2 e^{-\nu_2^* t}$, *with convergence rate* $\nu_2^* := 2(\alpha exp(\mu) + m_2)$, *larger than that of the full-order observer* (18)-(21).

In this section, we have thus described three different exponential observers for the plug flow reactor model. The first one (eq. (14)-(17)) is derived directly from our main result (Theorem 2.1). The second one (eq. (18)-(21)) shows that an exponential observer can be constructed even if the concentration is not measured and with only partial knowledge of the nonlinear part of the model. The third one (eq. (22)-(25)) improves the convergence rate of the concentration reconstruction error by reintroducing a measurement of the concentration.

Comment 2.2. *In (Aksikas et al., 2007), a result of asymptotic stability of the system Eq. 13 requires the following condition:*

$$k_N < \beta$$

In order to test the performance of the proposed observers, numerical simulations will be given when the above condition does not holds.

2.2 Simulation Results

Our objective is to illustrate the theoretical results related to the different exponential observers for the plug flow reactor model.
The process model has been initialized with two constant profiles $x_1(0,z) = -1$, and $x_2(0,z) = 0$. The observers have been initialized with $\hat{x}_1(0,z) = 0$, and $\hat{x}_2(0,z) = 1$. The equations have been integrated by using a backward finite difference approximation for the first-order space derivative $\partial/\partial z$

$$\frac{\partial x}{\partial z} \simeq \frac{x(t,z_i) - x(t,z_{i-1})}{\Delta z}$$

with $\Delta z = 1/100$.
In order to be close as possible to possibly unstable nonisothermal plug-flow reactor, we have selected the model (9)-(12) with $\beta = 0.2$. The adopted numerical values for the process parameters are taken from (Smets et al., 2002).

Table 1: Process parameters using for numerical simulations.

Process parameters	Numerical value
L	1 m
v	0.1 m/s
E	11250 cal/mol
k_0	$10^6 s^{-1}$
$\frac{4h}{\rho C_p d}$	$0.02 s^{-1}$
C_{in}	0.02 mol/L
R	1.986 cal/(mol.K)
T_{in}	340 K
$\frac{\Delta H}{\rho C_p}$	-4250 K.L/mol

Figure 1 shows the time evolution of the concentration error e_2 related to the exponential observer (14)-(17). Similar results are obtained for the two other observers.
In order to cover all the assumptions, the design parameter g related respectively to the exponential observer (14)-(17) and the exponential observer (18)-(21) has been taken respectively as $g = 2 * k_N$ and $g = 2 * \alpha\delta k_r$, and the design parameters m_1 and m_2 related to the exponential observer, (22)-(25) have

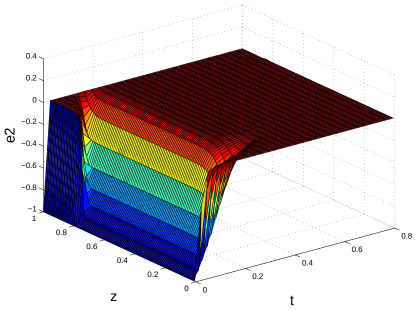


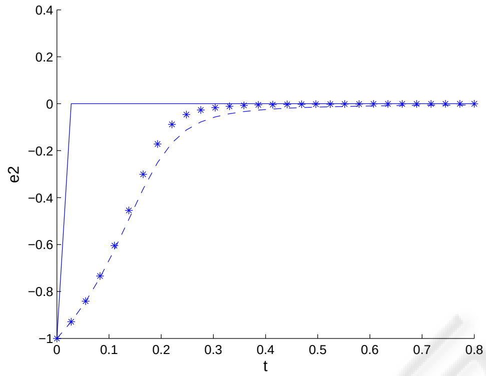
Figure 1: Evolution in time and space of the concentration error.

been taken as $m_1 = 2 * \alpha\delta k_r$ and $m_2 = 10 * m_1$, with $w = 3 * L/4$.

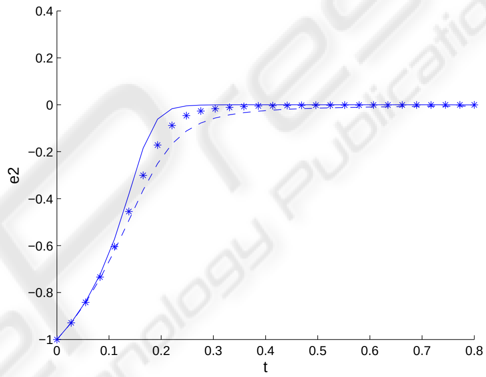
Figure 2 shows respectively the time evolution of the concentration error e_2 at the positions $0.1 * L$, $0.5 * L$ and $0.9L$, for the case where only the temperature is measured on the length interval $[3 * L/4, L]$ (the dashed line) i.e the exponential observer (18)-(21), and for the case where both the temperature and the concentration are measured on the same length interval with the exponential observer (14)-(17) (the solid line) and with the exponential observer (22)-(25) (the star line).
It is seen as expected that the concentration error related to the exponential observer (22)-(25) is faster than the one related to the exponential observer (18)-(21), however it remains slower than that related to the observer (14)-(17), which represents the ideal case, since in that case, the nonlinear part is assumed to be exactly known.

3 CONCLUSIONS

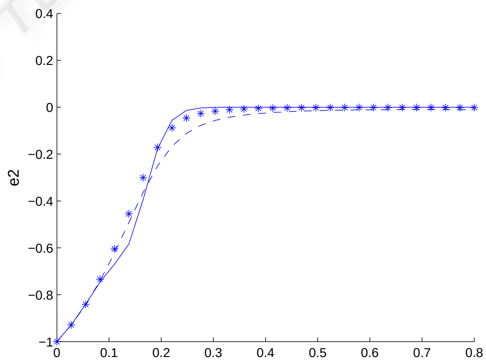
This paper presents sufficient conditions to construct an exponential observer for a nonlinear infinite dimensional system driven in a real Hilbert state description. The theory is applied to a non-isothermal plug flow tubular reactor governed by hyperbolic first order partial differential equations. Several observer structures are proposed, depending on the part of the states that are available for measurement and on the knowledge of the nonlinear part of the model. Performance issues of the different observer designs are illustrated by simulation results. The best performance is obviously obtained when the nonlinear term is perfectly known and both states (temperature and concentration) are measured at the end of the reactor. However, we also show that good results can be achieved when only the temperature is measured and when bounds on the nonlinear term are used in the observer dynamics. Finally, an improved convergence rate for the estimation error on the concentration can be obtained when re-introducing a measurement of the concentration at the end of the reactor. These observers include design parameters that can be tuned by the user to satisfy specific needs in terms of convergence rate.



(a) concentration error at z=0.9*L.



(b) concentration error at z=0.5*L.



(c) concentration error at z=0.1*L.

Figure 2: Convergence of the concentration error for the three proposed observers.

REFERENCES

Achhab, M. E., Laabissi, M., Winkin, J., and Dochain, D. (1999). State trajectory analysis of plug flow non-isothermal reactors using a nonlinear model. In *Proceedings of the 38th IEEE Conference on Decision and Control, Phoenix, Arizona, USA, 663-667, December*.

Aksikas, I., Winkin, J., and Dochain, D. (2007). Asymptotic stability of infinite-dimensional semilinear systems: application to a nonisothermal reactor. In *Systems and Control Letters, vol. 56, 122-132.*

Bounit, H. and Hammouri, H. (1997). Observers for infinite dimensional bilinear systems. European J. of Control, vol. 3, 325-339.

Cazenave, T. and Haraux, A. (1998). *An Introduction to Semilinear Evolution Equations*. Clarendon Press. Oxford.

Curtain, R. F. and Zwart, J. (1995). *An Introduction to Infinite Dimentional Linear Systems Theory*. Springer, New York.

Gauthier, J. P., Xu, C. Z., and Bounabat, A. (1995). Observers for infinite dimentional dissipative bilinear systems. In *J. Math. Systems, Estimation and Control, vol. 5, 119-122*.

Gressang, R. V. and Lamont, G. B. (1975). Observers for systems characterized by semigroups. In *IEEE Trans. Aut. Contr, vol. 20, 523-528*.

Laabissi, M., Achhab, M., Winkin, J., and Dochain, D. (2001). Trajectory analysis of nonisothermal tubular reactor nonlinear models. In *Systems and Control Letters, 42, 169-184*.

Orlov, Y. and Dochain, D. (2002). Discontinuous feedback stabilisation of minimum-phase semilinear infinite-dimensional systems with application to chemical tubular reactor models. In *IEEE Trans. Aut. Contr, vol. 47, 1293-1304*.

Pazy, A. (1983). *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer-Verlag.

Smets, I., Dochain, D., and Impe, J. V. (2002). Optimal temperature control of a steady-state exothermic plug-flow reactor. In *AIChE Journal, Vol. 48, No. 2, 279-286*.

Winkin, J., Dochain, D., and Ligarius, P. (2000). Dynamical analysis of distributed parameter tubular reactors. In *Automatica, 36, 349-361*.