A Spiking Neuron Model based on the Lambert W Function

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Abstract: A model of spiking neuron based on the Lambert W function has been proposed. It is shown analytical dependence of spiking neuron firing time on input spikes can be obtained. Though such dependence is rather complex, it still allows of simplifying software implementation of spiking neural networks. It is demonstrated the proposed model software implementation operates faster than one of straightforward propagation of a spike through multiple synapse and soma of spiking neuron.

1 INTRODUCTION

From the software implementation standpoint, of greater importance is analytical representation of spiking neuron firing time dependence on input spikes inasmuch as firing time holds a central position both in conventional self-learning spiking neural networks (as a parameter to determine cluster that the input pattern belongs to (Natschlaeger and Ruf, 1998)) and in hybrid systems based on them (as a distance between input pattern and clusters, that is utilized for fuzzy partitioning (Bodyanskiy and Dolotov, 2009)). Such dependence has not been obtained till now so, when implementing a software model of spiking neural network, a researcher has to emulate dynamics of spiking neuron soma membrane potential in order to determine empirically the moment when it crosses firing threshold, which differs radically from conventional artificial neural networks where a neuron output is readily expressed on its inputs. Considering population coding is usually used in self-learning spiking neural networks (Bohte et al., 2002), and their synapses are compound structures (Gerstner and Kistler, 2002) – so even one input gives rise to a set of spikes that come to soma via different paths, software applications based on spiking neural networks may operate significantly slowly because of necessity to emulate spiking neuron membrane potential dynamics.

In the next sections, it is shown that output spike firing time dependence on incoming spikes may be expressed analytically based on the Lambert W function and thus the mentioned difficulty in spiking neural networks implementation may be overcome. Performance of the proposed model of spiking neuron is compared with one of a straightforward model of spiking neuron.

2 ANALYTICAL DEPENDENCE OF SPIKING NEURON FIRING TIME ON INPUT SPIKES

In order to obtain an analytical dependence of spiking neuron firing time on input spikes, let us solve the simpler task first: obtaining firing time of a spiking neuron when it receives one incoming spike (we will use 'conventional' architecture of self-learning spiking neural network introduced in (Bohte et al., 2002)).

Spiking neuron receives input signal in a pulse-position form (incoming spikes), transforms it into continuous-time form (membrane potential), and transforms it back to pulse-position form on its output (outgoing spike). Let us examine such transformation using spiking neuron j with a simple (not multiple) synapse without time delay that connects the i-the neuron of the previous layer (it may be either a receptive neuron or a spiking neuron) with the neuron. Its membrane potential is

$$u_j(t) = w_{ji}\varepsilon(t - t_i), \tag{1}$$

$$\varepsilon(t - t_i) = \frac{t - t_i}{\tau}\exp\left(1 - \frac{t - t_i}{\tau}\right)H(\varepsilon(t - t_i)) \tag{2}$$

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where t_i is a spike produced by the i-th neuron, w_{ji} is a synaptic weight between the i-th and the j-th neurons, $\varepsilon(\bullet)$ is a spike-response function, τ is the membrane potential decay time constant, $H(\bullet)$ is the Heaviside step function. At the moment when $u_j(t)$ reaches firing threshold $\theta_{\text{s.n.}}$, the spiking neuron generates outgoing spike t_j on its output. The task is to find dependence $t_j(t_i)$.

In order to solve the problem, we have to utilize the function that is inverse to function

$$f(z) = ze^{z} \tag{3}$$

where z is a complex variable. Plot of function $f(z)$ is depicted on Figure 1.

Figure 1: Function $f(z) = ze^{z}$.

The inversion function of $f(z)$ is the Lambert W function, also called the omega function, $\Omega(z)$ (Corless, Gonnet, Hare, et al., 1996). It cannot be expressed in terms of elementary functions. It has two main branches on interval $\left[-\frac{1}{e}, 0\right)$ (Figure 2): $\Omega_{-1}(x)$ when $\Omega(x) < -1$ (dashed line) and $\Omega_0(x)$ when $\Omega(x) \geq -1$ (solid line).

Let us solve now the equation

$$u_j(t) = \theta_{\text{s.n.}} \tag{4}$$

for t (t is apparently less than simulation interval time t_{sim}). Using (1), (2), and the Heaviside step function definition, we can express membrane potential of the j-th neuron as follows:

$$u_j(t) = \begin{cases} w_{ji} \dfrac{t - t_i}{\tau} e^{1 - \frac{t - t_i}{\tau}}, & t \geq t_i; \\ 0, & t < t_i. \end{cases} \tag{5}$$

Case $t < t_i$ doesn't make sense as a spiking neuron can't fire until the only incoming spike t_i reaches it so the equation (4) takes from

$$w_{ji} \frac{t - t_i}{\tau} e^{1 - \frac{t - t_i}{\tau}} = \theta_{\text{s.n.}}\,. \tag{6}$$

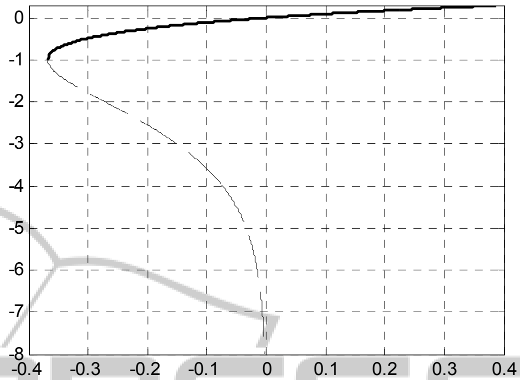

Figure 2: Function $\Omega(x)$.

It should be noted that, from practical considerations, parameters of equation (6) may be bounded as follows:

$$w_{ji} > 0\,, \tag{7}$$

$$\tau > 0\,, \tag{8}$$

$$\theta_{\text{s.n.}} > 0\,. \tag{9}$$

Now, applying definition of $\Omega_0(x)$ (as we aim to get time when membrane potential crosses firing threshold from below), we obtain

$$-\frac{t - t_i}{\tau} = \Omega_0\left(-\frac{\theta_{\text{s.n.}}}{e w_{ji}}\right) \tag{10}$$

or

$$t = t_j = t_i - \tau \Omega_0\left(-\frac{\theta_{\text{s.n.}}}{e w_{ji}}\right). \tag{11}$$

Let us consider now more complex case when two spikes t_1 and t_2 generated by neurons of the previous layer reach the j-th spiking neuron, and the first neuron in the previous layer has fired earlier than the second one, i.e.

$$t_1 < t_2\,. \tag{12}$$

Equation (4) will take the following form in such case:

$$\begin{aligned} & w_{j1} \frac{t - t_1}{\tau} e^{1 - \frac{t - t_1}{\tau}} H(t - t_1) + \\ & + \; w_{j2} \frac{t - t_2}{\tau} e^{1 - \frac{t - t_2}{\tau}} H(t - t_2) = \theta_{\text{s.n.}} \end{aligned} \tag{13}$$

or in an expanded form:

$$\left[\begin{array}{l}\begin{cases}0=\theta_{\text{s.n.}},\\ t<t_1,\, t<t_2;\end{cases}\\ \begin{cases}w_{j2}\dfrac{t-t_2}{\tau}e^{1-\frac{t-t_2}{\tau}}=\theta_{\text{s.n.}},\\ t<t_1,\, t\geq t_2;\end{cases}\\ \begin{cases}w_{j1}\dfrac{t-t_1}{\tau}e^{1-\frac{t-t_1}{\tau}}=\theta_{\text{s.n.}},\\ t\geq t_1,\, t<t_2;\end{cases}\\ \begin{cases}w_{j1}\dfrac{t-t_1}{\tau}e^{1-\frac{t-t_1}{\tau}}+w_{j2}\dfrac{t-t_2}{\tau}e^{1-\frac{t-t_2}{\tau}}=\theta_{\text{s.n.}},\\ t\geq t_1,\, t\geq t_2,\, t<t_{\text{sim}}.\end{cases}\end{array}\right. \tag{14}$$

Taking into account (9) and (12), the first and the second systems of equations from (14) do not make sense. Solution of the third system of equations is similar to (11), namely

$$t=t_j=t_1-\tau\Omega_0\left(-\frac{\theta_{\text{s.n.}}}{ew_{j1}}\right). \tag{15}$$

By solving the fourth system of equations from (14), we obtain the following dependence

$$t=\frac{w_{j1}t_1e^{\frac{t_1}{\tau}}+w_{j2}t_2e^{\frac{t_2}{\tau}}}{w_{j1}e^{\frac{t_1}{\tau}}+w_{j2}e^{\frac{t_2}{\tau}}}-\tau\times$$
$$\times\Omega_0\left(-\frac{\theta_{\text{s.n.}}}{e}\frac{e^{\frac{w_{j1}t_1e^{\frac{t_1}{\tau}}+w_{j2}t_2e^{\frac{t_2}{\tau}}}{\tau\left(w_{j1}e^{\frac{t_1}{\tau}}+w_{j2}e^{\frac{t_2}{\tau}}\right)}}}{w_{j1}e^{\frac{t_1}{\tau}}+w_{j2}e^{\frac{t_2}{\tau}}}\right). \tag{16}$$

It is notable that equation (16) is substantially generalization of equation (15): it defines a wave whose inverse form is identical to a separate postsynaptic potential and that considers effect of the preceding spike on the neuron's membrane potential.

3 A SPIKING NEURON MODEL

Let us generalize equation (16) now for case of arbitrary number of incoming spikes.

It is worthy of note that solution (15), (16) of equation (13) defines two time intervals – $[t_1,t_2)$ and $[t_2,t_{\text{sim}})$ where on each interval, membrane potential of spiking neuron's soma takes wave-like form that is caused by two incoming spikes. Utilizing $\Omega_0(x)$ in (15), (16), we consider on the mentioned interval only those lapses where membrane potential monotonically increases since we need to obtain moment when membrane potential reaches firing threshold from below. Solution (15), (16) gives firing time of neuron when its membrane potential reaches firing threshold either on the interval when its value monotonically increases for the first time (soma receives incoming spike t_1) or on the interval when it increases for the second time (soma receives incoming spike t_2). If solution (15) does not produce a real value, it means the membrane potential has not reached firing threshold still so the second interval should be analyzed. If solution (16) has not produces a real value either, it means two incoming spikes are not sufficient to fire the neuron.

Evidently the reasoning above may be applied to arbitrary number of incoming spikes so in order to obtain a generalized solution, we have to analyze each interval one by one where membrane potential increases to find the first moment when it reaches firing threshold. Such exhaustive search apparently requires much less number of comparisons as opposed to continual comparing on each time step in straightforward modelling of spiking neurons. To perform the comparison, all spikes incoming to the j-th neuron should be arranged in order of firing time magnitude (so the set of incoming spikes $T=\{t_{ji}\,|\,0\leq t_{ji}\leq t_{\max}\ \forall i,\, i=\overline{1,n}\}$ where $t_{\max}$ is the latest possible firing time of neuron of the previous layer, n is the number of neurons in the previous layer, should be transformed to linearly ordered set $\{t_{j\hat{i}}\,|\,t_{j,\hat{i}-1}\leq t_{j\hat{i}},\, t_{j\hat{i}}\in T\,\forall\hat{i},\hat{i}=\overline{1,n}\}$). Then the simulation interval should be broken down with respect to the ordered set of incoming spikes (intervals $[t_{j,1},t_{j,2}),[t_{j,2},t_{j,3}),\ldots[t_{j,\hat{i}-1},t_{j\hat{i}}),\ldots[t_{jn},t_{\text{sim}})$). Finally, each interval should be analyzed sequentially whether membrane potential reached firing threshold – once the first real value is obtained, the search should be stopped.

By increasing number of addends to current number of intervals that have been analyzed, we can generalize equations (15), (16) as follows:

$$t=t_j=\frac{A}{B}-\tau\Omega_0\left(-\frac{\theta_{\text{s.n.}}}{e}\frac{e^{\frac{A}{\tau B}}}{B}\right), \tag{17}$$

$$A=\sum_{\hat{k}=1}^{\hat{i}}t_{j\hat{k}}w_{j\hat{k}}e^{\frac{t_{j\hat{k}}}{\tau}}, \tag{18}$$

$$B = \sum_{\hat{k}=1}^{\hat{i}} w_{j\hat{k}} e^{\frac{t_{j\hat{k}}}{\tau}} . \quad (19)$$

where $\hat{i}$ is the number of interval currently being analyzed; $t_{j\hat{k}}$ is an enumerated spike, $w_{j\hat{k}}$ is a weight of synapse that spikes comes to soma through. Now if one calculates t_j according to (17)-(19) on each time interval until a real value is received, he can obtain spiking neuron firing time for an arbitrary number of incoming spikes (Figure 3 illustrates case with three incoming spikes).

Thus, having analytical model of spiking neuron, a researcher can easily implement a software application of self-learning spiking neural network (learning procedures of spiking neural networks are out of scope of this paper). Under easy software implementation, we understand the fact that a researcher does not have to program spike propagation form a receptive neuron or a spiking neuron through multiple synapse to soma of the spiking neuron whose firing time is being obtained. In a sense, the proposed model of spiking neuron is akin to conventional models of artificial neural networks of the second generation as they are constructed in terms of matrix algebra, thus allowing developers and researcher to avoid biological aspect of neurons operating.

An additional advantage of the proposed model is that it can operate in a sequence mode when new spikes constantly come to spiking neuron inputs. However, we have to note here that spiking neuron refractoriness and effect of spike-after potential on further neuron firing are not considered in this work as in any case they play no part in the most of spiking neural networks used in actual practice.

4 SPIKING NEURON SOFTWARE IMPLEMENTATIONS PERFORMANCE

Nowadays software applications of the designed models and systems are in most common use due to their simplicity and low price as compared to hardware implementations. This brings up an important question on performance of different spiking neuron software implementations. Surprisingly, ways to improve spiking neural network models for software implementation are poorly researched. This section describes results of performance testing of two spiking neuron software implementations – straightforward model and the model introduced in this paper based on the Lambert W function.

The straightforward model of spiking neuron (an example of it can be found in (De Berredo, 2005)) emulates spiking neuron membrane potential dynamics and has to check whether its value crossed firing threshold on each time step.

Software implementation of the spiking neuron model introduced on the Lambert W function base rests on the procedure described in the previous section: incoming spikes are put in order of their firing time magnitude and t_j is calculated with (17)-(18) on each time interval formed; the first real value of t_j indicates firing time of spiking neuron.

As seen from Table 1, the introduced model is always faster than the straightforward model though its operating time raises as size of input spikes vector increases.

Table 1: Results of performance testing of straightforward model of spiking neuron and the model proposed in this paper.

Size of input spikes vector	Firing time calculation speed, s	
	Straightforward model	Model based on the Lambert W function
10	0.0019	0.0012
50	0.254	0.148
100	0.706	0.308
300	2.129	1.524
500	4.018	1.757

We have to note here that in practice, a range of various techniques are used to improve performance of software implementations (e.g., methods of matrix algebra). We used just ‘pure’ models for the sake of reference models comparison.

5 CONCLUSIONS

The major conclusion of the research is that analytical dependence of spiking neuron firing time on input spikes can be expressed – but in an intricate way. That fact complicates comprehensive analysis of spiking neural networks behavior and features. However, the proposed spiking neuron model allows of improving spiking neural networks software implementations performance. It also allows a researcher to abstract away from biological specific of spiking neural networks when implementing them and to use them just as a regular tool for data processing. Another advantage of the proposed model is its precision level of the firing time

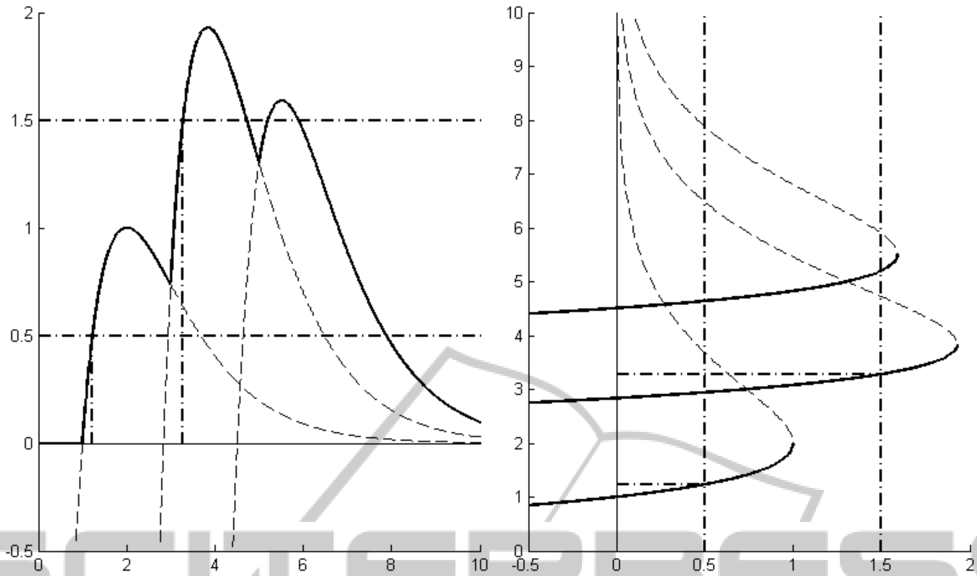

Figure 3: Spiking neuron firing time obtaining in case of three incoming spikes. Left: Spiking neuron soma membrane potential dynamics caused by incoming spikes (solid line) is presented as a sum of single postsynaptic potentials defined by spike-response function (2) (dashed line). Two different firing thresholds are marked with horizontal dash-and-dot lines. In the straightforward model of spiking neuron, neuron firing event occurrence is checked on each sampled time step. Sampled time when that event occurred defines t_j (vertical dash-and-dot line). Right: Dependence of t_j on firing threshold indicates equation (4) solution based on $\Omega_0(x)$ (solid line) and $\Omega_{-1}(x)$ (dashed line). Considered firing threshold values are marked with vertical dash-and-dot lines. In the model based on the Lambert W function, there may be three checks at most to obtain t_j. Given firing threshold is 0.5, firing time is obtained on the first step (bottom horizontal dash-and-dot line), and given firing threshold is 1.5, firing time is obtained on the second step (top horizontal dash-and-dot line).

calculation that is important in fuzzy spiking neural networks: contrary to the straightforward model where precision is bounded with the sampled interval value, precision in the proposed model is bounded only by precision of the system in the Lambert W function calculation.

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