Sensitivity Estimation by Monte-Carlo Simulation Using Likelihood Ratio Method with Fixed-Sample-Path Principle

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Abstract: The likelihood ratio method (LRM) is an efficient indirect method for estimating the sensitivity of given expectations with respect to parameters by Monte-Carlo simulation. The restriction on application of LRM to real-world problems is that it requires explicit knowledge of the probability density function (pdf) to calculate the score function. In this study, a fixed-sample-path method is proposed, which derives the score function required for LRM not via the pdf but directly from a constructive algorithm that computes the sample path from parameters and random numbers. The boundary residual, which represents the correction associated with the change of the distribution range of the random variables in LRM, is also derived. Some examples including the estimation of risk measures (Greeks) of option and financial flow-of-funds networks showed the effectiveness of the fixed-sample-path method.

1 INTRODUCTION

Given a system of interest, it is a major concern for engineers and designers to understand how to make the system behaviour "desirable" by changing parameters. To this end, knowledge about the relationship (or the sensitivity) between the system parameters and the system behaviours is required. However, for complicated and probabilistic systems, the relationship between parameters and system behaviour is often unclear, so Monte-Carlo simulation is needed to estimate the relation.

Let X denote a random variable describing system behaviours under consideration and x denote its sample value (sample path). Here, X can be a multi-dimensional vector and/or a family of random variables indexed by "time" t (random process). Therefore, X should be denoted as $\mathbb{X}(t)$ in nature, but we henceforth use X to avoid cumbersome notation. Assume X is dependent on the system parameters $\mathbb{z}$, where $\mathbb{z} = (z_i)_{i=1\ldots N}$ is an N-dimensional vector, and let $f(x, \mathbb{z})$ be the probability density function (pdf) of X. There exists a behaviour evaluation function $a(x)$ that maps the system behaviour x to a real value $a(x)$. We call the expectation value $A(\mathbb{z})$ of $a(X)$, i.e.

$$A(\mathbb{z}) = \mathbb{E}[a(X)] = \int a(x)\, f(x, \mathbb{z})\, dx \tag{1}$$

a "system evaluation function".

To calculate $A(\mathbb{z})$ of real interesting systems, the fact that density function $f(x, \mathbb{z})$ is often unknown becomes an obstacle. However, even if $f(x, \mathbb{z})$ itself is unknown, in many cases, the system behaviour, which makes the distribution of X, is known and modelled, and Monte-Carlo simulation is applicable.

Let $\mathbb{W} = \left(W_j\right)_{j=1\ldots M}$ denote an M-dimensional vector of random numbers and $\mathbb{w} = \left(w_j\right)_{j=1\ldots M}$ denote its sample value. We suppose that the simultaneous probability density function $g(\mathbb{w})$ of $\mathbb{W}$ is well-known, and we can easily generate random numbers of this distribution on computers. Examples of $\mathbb{W}$ include M number of independent random numbers from a uniform distribution on $(0,1)$ or the standard normal distribution $N(0,1)$. Assuming that there exists a function $x = x(\mathbb{w}, \mathbb{z})$, i.e., a constructive algorithm to compute x from the parameters $\mathbb{z}$ and random numbers $\mathbb{w} = \left(w_j\right)_{j=1\ldots M}$,

$$A(\mathbb{z}) = \mathbb{E}[a(X)] = \int_{\Omega} a(x(\mathbb{w}, \mathbb{z}))g(\mathbb{w})\, d\mathbb{w} \tag{2}$$

holds, where $\Omega \subset \mathbb{R}^M$ denotes the support of $g(\mathbb{w})$. Using the L set of random numbers $\left(\mathbb{w}^{(k)}\right)_{k=1\ldots L} = \left(w_j^{(k)}\right)_{j=1\ldots M, k=1\ldots L}$, we can estimate $A(\mathbb{z})$ by

$$A(\mathbb{z}) \cong \frac{1}{L}\sum_{k=1}^{L} a\left(x\left(\mathbb{w}^{(k)}, \mathbb{z}\right)\right). \tag{3}$$

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Now, our goal is to estimate the sensitivity

$$\frac{\partial \mathrm{A}(\mathbb{z})}{\partial \mathbb{z}} = \left(\frac{\partial \mathrm{A}(\mathbb{z})}{\partial \mathrm{z_i}}\right)_{\mathrm{i}=1\ldots\mathrm{N}} \quad (4)$$

of the behaviour evaluation function with respect to all of the N numbers of parameters $\mathbb{z}$ by Monte-Carlo simulation. A direct method to this end is the finite differential method (FDM), which re-runs a set of Monte-Carlo simulations under a small variation Δz_i of the parameter z_i and approximates

$$\begin{aligned}\frac{\partial \mathrm{A}(\mathbb{z})}{\partial \mathrm{z_i}} &\cong \frac{\mathrm{A}(\mathbb{z}+\Delta \mathrm{z_i}) - \mathrm{A}(\mathbb{z})}{\Delta \mathrm{z_i}} \\ &\cong \frac{1}{\Delta \mathrm{z_i}}\left[\frac{1}{\mathrm{L}}\sum_{\mathrm{k}=1}^{\mathrm{L}} \mathrm{a}\left(\mathrm{x}\left(\mathbb{w}'^{(\mathrm{k})}, \mathbb{z}+\Delta \mathrm{z_i}\right)\right) \right. \\ &\qquad \left. - \frac{1}{\mathrm{L}}\sum_{\mathrm{k}=1}^{\mathrm{L}} \mathrm{a}\left(\mathrm{x}\left(\mathbb{w}^{(\mathrm{k})}, \mathbb{z}\right)\right)\right].\end{aligned} \quad (5)$$

where $\mathbb{z} + \Delta z_i = (z_1, z_2, \ldots, z_i + \Delta z_i, \ldots, z_N)$. The problem of FDM is that the convergence speed of Eq. (5) is slow. The variance of estimation value $\mathrm{A}(\mathbb{z})$ by Eq. (3) is proportional to L^{-1}, whereas that of estimation value $\frac{\partial \mathrm{A}(\mathbb{z})}{\partial z_i}$ by Eq. (5) is proportional to $\mathrm{L}^{-1/4}$ (for indepent sampling) or $\mathrm{L}^{-1/3}$ (for common sampling) (Glynn, 1989).Moreover, since the re-run of a set of Monte-Carlo simulations is needed for each parameter z_i, the computational time might be impractical if the number of parameters N is large.

There exist two known indirect methods for estimating the sensitivity more efficiently than FDM: the pathwise derivative method (PDM) (Glasserman , 2003, Rubinstein and Kroese, 2007, Ho and Cao, 1991, Bettonvil, 1981) and the likelihood ratio method (LRM) (Glasserman, 2003, Rubinstein and Kroese, 2007, Glynn, 1987). PDM (also called the "infinitesimal perturbation method") is based on the idea of differentiating Eq. (2) with respect to the parameters $\mathbb{z}$,

$$\begin{aligned}\frac{\partial \mathrm{A}(\mathbb{z})}{\partial \mathbb{z}} &= \frac{\partial}{\partial \mathbb{z}}\int_{\Omega} \mathrm{a}(\mathrm{x}(\mathbb{w},\mathbb{z}))\mathrm{g}(\mathbb{w})\,\mathrm{d}\mathbb{w} \\ &= \int_{\Omega} \frac{\partial \mathrm{a}(\mathrm{x}(\mathbb{w},\mathbb{z}))}{\partial \mathbb{z}}\mathrm{g}(\mathbb{w})\,\mathrm{d}\mathbb{w}.\end{aligned} \quad (6)$$

Assuming this holds, we estimate the sensitivity by

$$\frac{\partial \mathrm{A}(\mathbb{z})}{\partial \mathbb{z}} \cong \frac{1}{\mathrm{L}}\sum_{\mathrm{k}=1}^{\mathrm{L}} \frac{\partial \mathrm{a}(\mathrm{x}(\mathbb{w},\mathbb{z}))}{\partial \mathbb{z}}. \quad (7)$$

PDM is quite effective in term of its small variance and fast convergence speed. However, it has limited range of application because interchanging the order of integration and differentiation in Eq. (6) requires that $\mathrm{a}(\mathrm{x}(\mathbb{w},\mathbb{z}))$ is (almost surely) continuous with respect to $\mathbb{z}$, which is often not the case.

In comparison, LRM (also called the "score function method") is based on the idea of differentiating Eq. (1) with respect to the parameters $\mathbb{z}$,

$$\begin{aligned}\frac{\partial \mathrm{A}(\mathbb{z})}{\partial \mathbb{z}} &= \frac{\partial}{\partial \mathbb{z}}\int \mathrm{a}(\mathrm{x})\,\mathrm{f}(\mathrm{x},\mathbb{z})\,\mathrm{dx} \\ &= \int \mathrm{a}(\mathrm{x})\frac{\partial \mathrm{f}(\mathrm{x},\mathbb{z})}{\partial \mathbb{z}}\mathrm{dx} \\ &= \int \mathrm{a}(\mathrm{x})\,\mathrm{h}(\mathrm{x},\mathbb{z})\,\mathrm{f}(\mathrm{x},\mathbb{z})\,\mathrm{dx}\,,\end{aligned} \quad (8)$$

where

$$\mathrm{h}(\mathrm{x},\mathbb{z}) = \frac{\partial \mathrm{f}(\mathrm{x},\mathbb{z})}{\partial \mathbb{z}}\Big/\mathrm{f}(\mathrm{x},\mathbb{z}) = \frac{\partial \log[\mathrm{f}(\mathrm{x},\mathbb{z})]}{\partial \mathbb{z}} \quad (9)$$

is called a "score function". Assuming this holds, we estimate the sensitivity by

$$\frac{\partial \mathrm{A}(\mathbb{z})}{\partial \mathbb{z}} \cong \frac{1}{\mathrm{L}}\sum_{\mathrm{k}=1}^{\mathrm{L}} \mathrm{a}\left(\mathrm{x}\left(\mathbb{w}^{(\mathrm{k})},\mathbb{z}\right)\right)\,\mathrm{h}\left(\mathbb{w}^{(\mathrm{k})},\mathbb{z}\right). \quad (10)$$

LRM has a wider range of application than does PDM because the pdf $\mathrm{f}(\mathrm{x},\mathbb{z})$ is typically a smooth function with respect to the parameters $\mathbb{z}$, whereas $\mathrm{a}(\mathrm{x}(\mathbb{w},\mathbb{z}))$ is not. An exception that does not satisfy Eq. (8) will be discussed later in Section 2.3.

The restriction on the application of LRM to real systems is that it requires explicit knowledge of the pdf $\mathrm{f}(\mathrm{x},\mathbb{z})$ to calculate the score function $\mathrm{h}(\mathrm{x},\mathbb{z})$ from Eq. (9). This restriction might seem not to be a problem because we know the pdf of the random numbers $\mathrm{g}(\mathbb{w})$ and the constructive algorithm, which computes the sample path $\mathrm{x} = \mathrm{x}(\mathbb{w},\mathbb{z})$ from $\mathbb{w}$, and $\mathrm{f}(\mathrm{x},\mathbb{z})$ can be calculated from these in theory. In fact, pdf $\mathrm{f}(\mathrm{x},\mathbb{z})$ can be decomposed to the products of some conditional probability density functions for some systems, such as Markov chains, and discrete event systems without agent loop (Glasserman, 2003, Rubinstein and Kroese, 2007). Nevertheless, considering that we apply Monte-Carlo simulation due to the lack of explicit knowledge on the pdf $\mathrm{f}(\mathrm{x},\mathbb{z})$, the derivation of the score function from Eq. (9) is an intrinsically problematic approach.

In this study, we propose a "fixed-sample-path" method. Using this method, we can derive the score function not via the pdf $\mathrm{f}(\mathrm{x},\mathbb{z})$ but directly from the constructive algorithm that computes the sample path $\mathrm{x} = \mathrm{x}(\mathbb{w},\mathbb{z})$ from the parameters $\mathbb{z}$ and the sample values $\mathbb{w}$ of the random numbers.

The paper is organized as follows. In Section 2, we describe the idea of the fixed-sample-path method and its formulation. In Section 3, the fixed-sample-path method is applied to two simple systems: a system with 2-dimensional uniform random variables and the estimation of risk measures (Greeks) of option pricing in finance. Section 4 is a description of a more complicated example: a financial flow-of-funds network of 25 companies. Finally, Section 5 is the conclusion.

2 LRM WITH FIXED-SAMPLE-PATH PRINCIPLE

2.1 Basic Idea

The reason the calculation of the score function $h(x,\mathbb{z})$ requires explicit knowledge of the pdf $f(x,\mathbb{z})$ is that LRM in Eq. (8) is derived by differentiating Eq. (1), which depends on $f(x,\mathbb{z})$. In comparison, Eq. (2), which is used to derive PDM in Eq. (6), only depends on $g(\mathbb{w})$ and $x(\mathbb{w},\mathbb{z})$, which we know explicitly. Can we not derive LRM from Eq. (2) instead of Eq. (1)?

Let us consider sensitivity with respect to the i-th parameter z_i. The key idea is, given a sample path $x = x(\mathbb{w},\mathbb{z})$ and a small variation Δz_i of the parameter z_i, to consider the small variation $\Delta\mathbb{w}$ of the sample values $\mathbb{w}$ of random variables that cancels the parameter variation, i.e, $\Delta\mathbb{w}$ satisfying

$$A(\mathbb{z}) = \mathbb{E}[a(X)] = \int_\Omega a(x(\mathbb{w},\mathbb{z}))g(\mathbb{w})\,d\mathbb{w} \qquad (11)$$

Since $\mathbb{w}+\Delta\mathbb{w}$ is, of course, not distributed with pdf $g(\mathbb{w})$, the expectaion $A(\mathbb{z}+\Delta z_i)$ is no longer calculable with simple expectation Eq. (3). However, using the importance sampling method, we can estimate $A(\mathbb{z}+\Delta z_i)$ by averaging up $x(\mathbb{w},\mathbb{z}) = x(\mathbb{w}+\Delta\mathbb{w},\mathbb{z}+\Delta z_i)$ with appropriate weights.

Concretely speaking, given a sample path $x = x(\mathbb{w},\mathbb{z})$, we consider a "fixed-sample-path" derivative of the random variables $\mathbb{w}$ with respect to the parameter z_i under the condition of fixing the sample path x:

$$\left.\frac{\partial\mathbb{w}}{\partial z_i}\right|_{x=\mathrm{const}} = -\frac{\frac{\partial x}{\partial z_i}}{\frac{\partial x}{\partial \mathbb{w}}}. \qquad (12)$$

We note that the right-hand side of Eq. (12) is a formal expression because x might be a multi-dimensional vector. As discussed later in Section 2.3, the fixed-sample-path derivative $\left.\frac{\partial\mathbb{w}}{\partial z_i}\right|_{x=\mathrm{const}}$ can be calculated relatively easily from the constructive algorithm of the function $x = x(\mathbb{w},\mathbb{z})$. We use $\left.\frac{\partial\mathbb{w}}{\partial z_i}\right|_{x=\mathrm{const}}$ to calculate the score function $h(x,\mathbb{z})$.

2.2 Formulation of LRM

Let Δz_i be a small variation of the i-th parameter z_i. Then, the expectation $A(\mathbb{z}+\Delta z_i)$ under the parameter values $\mathbb{z}+\Delta z_i = (z_1, z_2, \dots, z_i+\Delta z_i, \dots, z_N)$ is

$$\begin{aligned}
&A(\mathbb{z}+\Delta z_i)\\
&= \int_\Omega a(x(\mathbb{w},\mathbb{z}+\Delta z_i))g(\mathbb{w})\,d\mathbb{w}\\
&= \int_{\Omega'} a\left(x\left(\mathbb{w}'+\Delta\mathbb{w},\mathbb{z}+\Delta z_i\right)\right)\frac{g(\mathbb{w}'+\Delta\mathbb{w})}{\left|\frac{d\mathbb{w}'}{d\mathbb{w}}\right|}\,d\mathbb{w}'\\
&= \int_{\Omega'} a(x)\frac{g(\mathbb{w}'+\Delta\mathbb{w})}{\left|\frac{d\mathbb{w}'}{d\mathbb{w}}\right|}\,d\mathbb{w}'\\
&= \int_{\Omega'} a(x)\frac{g(\mathbb{w}+\Delta\mathbb{w})}{\left|\mathbb{I}-\frac{d}{d\mathbb{w}}\left(\left.\frac{\partial\mathbb{w}}{\partial z_i}\right|_{x=\mathrm{const}}\right)\Delta z_i\right|}\,d\mathbb{w}\\
&\cong \int_{\Omega'} a(x)\left\{g(\mathbb{w})+\frac{\partial g}{\partial\mathbb{w}}\cdot\left[\left(\left.\frac{\partial\mathbb{w}}{\partial z_i}\right|_{x=\mathrm{const}}\right)\right]\Delta z_i\right\}\\
&\qquad \left\{1+\mathrm{tr}\left[\frac{d}{d\mathbb{w}}\left(\left.\frac{\partial\mathbb{w}}{\partial z_i}\right|_{x=\mathrm{const}}\right)\right]\Delta z_i\right\}d\mathbb{w},
\end{aligned} \qquad (13)$$

where $\left.\frac{\partial\mathbb{w}}{\partial z_i}\right|_{x=\mathrm{const}}$ is the ratio between the small parameter variation Δz_i and the small variation $\Delta\mathbb{w}$ of the random variable, which keeps x constant, i.e. $x = x(\mathbb{w},\mathbb{z}) = x(\mathbb{w}+\Delta\mathbb{w},\mathbb{z}+\Delta z_i)$ holds. Here, $\mathbb{I}$ is an identity matrix, tr denotes the trace of a matrix, and a centered dot "·" denotes the inner products of vectors. Also, $\left|\frac{d\mathbb{w}'}{d\mathbb{w}}\right|$ is the Jacobian determinant corresponding to the change of variables from $\mathbb{w}$ to $\mathbb{w}' = \mathbb{w}-\Delta\mathbb{w}$, and Ω' is the image of Ω under this transform. To get from line 5 to line 6, we use the following relation. For a matrix $B(\varepsilon) = (b_{ij}) = \mathbb{I} - \varepsilon D$ with a matrix $D = (d_{ij})$ and a small real number $\varepsilon \ll 1$, the first-order Taylor approximation around $\varepsilon = 0$ leads to

$$\begin{aligned}
\frac{1}{|\mathbb{I}-\varepsilon C|} &= \frac{1}{|B(\varepsilon)|}\\
&\cong \frac{1}{|B(0)|} - \varepsilon|B(0)|\sum_{i,j}\left(\left.c_{ji}\right|_{\varepsilon=0}\right)\left(\left.\frac{\partial b_{ij}}{\partial\varepsilon}\right|_{\varepsilon=0}\right)\\
&= \frac{1}{|\mathbb{I}|} - \varepsilon|\mathbb{I}|\sum_{i,j}\delta_{ji}\left(-d_{ij}\right)\\
&= 1+\varepsilon\,\mathrm{tr}(C),
\end{aligned} \qquad (14)$$

where (c_{ij}) is the inverse matrix of $B(\varepsilon)$ and (δ_{ij}) is an identity matrix.

Using Eq. (13), we obtain

$$\begin{aligned}\frac{\partial A}{\partial z_i} &= \lim_{\Delta z_i \to 0} \frac{A(\mathbb{z} + \Delta z_i) - A(\mathbb{z})}{\Delta z_i} \\ &= \int_\Omega a(x) \sum_{j=1}^{M} \left[\frac{\partial}{\partial w_j} \left(\frac{\partial w_j}{\partial z_i}\bigg|_{x=\mathrm{const}} \right) g(\mathbb{w}) + \frac{\partial g}{\partial w_j} \left(\frac{\partial w_j}{\partial z_i}\bigg|_{x=\mathrm{const}} \right) \right] d\mathbb{w} - R_i \\ &= \int_\Omega a(x)\, h_i(\mathbb{w}, \mathbb{z})\, g(\mathbb{w})\, d\mathbb{w} - R_i \\ &= \mathbb{E}[a(X)\, h_i(\mathbb{W}, \mathbb{z})\,] - R_i \,.\end{aligned} \tag{15}$$

Therefore, the score function $h_i(\mathbb{w}, \mathbb{z})$ can be written as

$$h_i(\mathbb{w}, \mathbb{z}) = \sum_{j=1}^{M} \left[\frac{\partial}{\partial w_j} \left(\frac{\partial w_j}{\partial z_i}\bigg|_{x=\mathrm{const}} \right) + \frac{\partial}{\partial w_j} \left(\frac{\partial w_j}{\partial z_i}\bigg|_{x=\mathrm{const}} \right) \right] \tag{16}$$

In addition, R_i is a term that is equivalent to the correction amount associated with the changing integration range Ω' of Eq. (13) to Ω. We call R_i a "boundary residual". From Eq. (13) and Eq. (15), we obtain

$$\begin{aligned}R_i &= \lim_{\Delta z_i \to 0} \frac{1}{\Delta z_i} \left\{ \int_{\Omega - \Omega'} a(x)\{1 + h_i(\mathbb{w}, \mathbb{z}) \Delta z_i\}\, g(\mathbb{w})\, d\mathbb{w} - \int_{\Omega' - \Omega} a(x)\{1 + h_i(\mathbb{w}, \mathbb{z}) \Delta z_i\}\, g(\mathbb{w})\, d\mathbb{w} \right\} \\ &= \int_{\partial\Omega} a(x)\, g(\mathbb{w}) \left(\frac{\partial \mathbb{w}}{\partial z_i}\bigg|_{x=\mathrm{const}} \right) \cdot \mathbb{m}\, d\mathbb{w} \\ &= \int_\Omega \mathrm{div} \left[a(x)\, g(\mathbb{w}) \left(\frac{\partial \mathbb{w}}{\partial z_i}\bigg|_{x=\mathrm{const}} \right) \right] d\mathbb{w} \\ &= \mathbb{E} \left[\frac{1}{g(\mathbb{W})} \mathrm{div} \left[a(x)\, g(\mathbb{w}) \left(\frac{\partial \mathbb{w}}{\partial z_i}\bigg|_{x=\mathrm{const}} \right) \right] \right]\end{aligned} \tag{17}$$

where $\Omega' - \Omega$ and $\Omega - \Omega'$ are the differential sets, $\partial\Omega$ is the boundary of Ω, $\mathbb{m}$ is the outward pointing unit vector of $\partial\Omega$, and div denotes the divergence with respect to $\mathbb{w}$. From Eq. (10), if $\frac{\partial \mathbb{w}}{\partial z_i}\Big|_{x=\mathrm{const}}$ is zero (vector) on the boundary $\partial\Omega$, then

$$\begin{aligned}\frac{\partial A}{\partial z_i} &= \int_\Omega a(x)\, h_i(\mathbb{w}, \mathbb{z})\, g(\mathbb{w})\, d\mathbb{w} \\ &= \mathbb{E}[a(X)\, h_i(\mathbb{W}, \mathbb{z})]\end{aligned} \tag{18}$$

holds.

Here, we calculate the score function $h_i(\mathbb{w}, \mathbb{z})$ for the typical distributions $g(\mathbb{w})$ of random variables with Eq. (16) for future convenience. For random variables $\mathbb{w}$ following the M-dimensional uniform distributions, considering $\frac{\partial g}{\partial w_j} = 0$, we obtain

$$h_i(\mathbb{w}, \mathbb{z}) = \sum_{j=1}^{M} \frac{\partial}{\partial w_j} \left(\frac{\partial w_j}{\partial z_i}\bigg|_{x=\mathrm{const}} \right). \tag{19}$$

For random variables $\mathbb{w}$ following the M-dimensional independent standard normal distributions, considering that pdf is written as $g(\mathbb{w}) = \prod_{j=1}^{M} g_N(w_j)$, where $g_N(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ is the pdf of (1-dimensional) standard normal distributions and $\frac{g_N'(x)}{g_N(x)} = -w$ holds, we obtain

$$h_i(\mathbb{w}, \mathbb{z}) = \sum_{j=1}^{M} \left[\frac{\partial}{\partial w_j} \left(\frac{\partial w_j}{\partial z_i}\bigg|_{x=\mathrm{const}} \right) - w_j \left(\frac{\partial w_j}{\partial z_i}\bigg|_{x=\mathrm{const}} \right) \right]. \tag{20}$$

Once the score function $h_i(\mathbb{w}, \mathbb{z})$ is calculated, we can estimate the sensitivity $\partial A / \partial z_i$ by Monte-Carlo simulation by using the LRM method:

$$\begin{aligned}\frac{\partial A}{\partial z_i} &= \mathbb{E}[a(X)\, h_i(\mathbb{W}, \mathbb{z})\,] - R_i \\ &\cong \frac{1}{L} \sum_{k=1}^{L} a\left(x(\mathbb{w}^k, \mathbb{z})\right) h_i(\mathbb{w}^k, \mathbb{z}) - R_i\end{aligned} \tag{21}$$

2.3 Discussion

The range in application of the fixed-sample-path method depends on the existance (and computability) of the fixed-sample-path derivative $\frac{\partial \mathbb{w}}{\partial z_i}\Big|_{x=\mathrm{const}}$. In other words, given a sample path x and a small variation Δz_i of the parameter z_i, the existance of the small variation $\Delta\mathbb{w}$ of the sample values $\mathbb{w}$ of random variables that satisfies Eq. (11) is the key to the fixed-sample-path method. This in general is not necessarily the case because the dimension of x, which we must keep fixed, can be bigger than the dimension of the random variables $\mathbb{w}$. Nevertheless, for many applications, especially for the case in which the system behaviour X is a time series $\mathbb{X}(t)$, the fixed-sample-path mathod is applicable. Here, we exemplify two cases. The first is the case in which the system behaviour $\mathbb{X}(t)$ can be written as

$$\mathbb{X}(t) = f(t, \mathbb{y}(\mathbb{w}, \mathbb{z}))\,, \tag{22}$$

i.e., $\mathbb{X}(t)$ follows a deterministic function $f(t, \mathbb{y})$ identified by a random variable $\mathbb{y}$ of which distribution is determined by the parameter $\mathbb{z}$. Clearly,

$$\frac{\partial \mathbb{w}}{\partial \mathbb{z}}\bigg|_{\mathbb{X}(t)=\mathrm{const}} = \frac{\partial \mathbb{w}}{\partial \mathbb{z}}\bigg|_{\mathbb{y}=\mathrm{const}} \tag{23}$$

holds in this case. The second is the case in which the time evolution of the system behaviour $\mathbb{X}(t)$ is detemined by the relation

$$\mathbb{X}(t+1) = f(\mathbb{X}(0), \mathbb{X}(1), \ldots, \mathbb{X}(t-1), \mathbb{W}^{(t)}, \mathbb{z})\,, \tag{24}$$

i,e., $\mathbb{X}(t+1)$ at $t+1$ is determined by the past history of $\mathbb{X}$ before t, a random variable $\mathbb{W}^{(t)}$ that is newly generated at each time t, and the parameter $\mathbb{z}$. In this case, given a small variation Δz_i of the parameter, the small variation $\Delta\mathbb{W}^{(t)}$ of the random variables that cancells out Δz_i can be determined sequencially from the initial time $t = 0$ to the ending time $t = T$ from the relation

$$\begin{aligned} &f(X(0), X(1), \ldots, X(t-1), \mathbb{W}^{(t)}, \mathbb{z}) \\ &\quad = f(X(0), X(1), \ldots, X(t-1), \mathbb{W}^{(t)} + \Delta\mathbb{W}^{(t)}, \mathbb{z} + \Delta\mathbb{z}) \,. \end{aligned} \tag{25}$$

Another point we want to note is the boundary residual R_i. An exception that the conventional LRM with Eq. (8) is not applicable includes the case where the integral range of Eq. (1) depends on the parameter z, i.e., the distribution range of the system behaviour X varies depending on the parameter z. The boundary residual R_i explicitly represents the correction amount associated with the change of the distribution range of X, and we can extend the range of application of LRM to this case using R_i as seen in Eq. (21). However, there would be considerable difficulty in numerical calculation of R_i by (17). Numerically efficient method for calculating R_i is one of the future works of this study.

3 EXAMPLE CALCULATIONS

In this section, we apply the calculation method proposed in section 2 to simple examples.

3.1 2-Dim. Uniform Distribution

As the first example, we consider a simple system consisting of a single parameter, $N = 1$, and two random variables, $M = 2$. Let $\mathbb{w} = (w_1, w_2)$ be 2-dimensional uniform random numbers on $(0,1) \times (0,1)$, i.e., $g(w_1, w_2) = 1$ on $\Omega = \{(w_1, w_2);\ 0 < w_1, w_2 < 1\}$, and otherwise, $g(w_1, w_2) = 0$, and let z be a real-valued parameter. Assuming the system behavior as $\mathbb{x} = (x_1, x_2) = (\sin(4z\, w_1),\ (2\,z + w_2)^2 + w_1)$ and the behavior evaluation function as $a(\mathbb{x}) = x_1 + x_2$, let us consider the sensitivity of the parameter value z to the expectation $A(z) = \mathbb{E}[a(\mathbb{x})]$. Note that since the distribution range of $x_2 = (2\,z + w_2)^2 + w_1$ depends on the parameter z, the boundary residual R is, as we look later, not zero for this example.

3.1.1 Direct Calculation

We can easily calculate the expectation $A(z)$ and its sensitivity $A'(z)$ by direct integration:

$$\begin{aligned} A(z) &= \iint_{\Omega} \{\sin(4z\, w_1) + z^2\, w_2 + w_1\}\, dw_1\, dw_2 \\ &= \frac{\sin^2(2z)}{2z} + 4z^2 + 2z + \frac{5}{6}\,, \\ A'(z) &= \frac{4z\sin(4z) + \cos(4z) + 8z^2(4z+1) - 1}{4z^2}. \end{aligned} \tag{26}$$

3.1.2 Calculation with Fixed Sample Method

Let us calculate $A'(z)$ by using LRM with the score function calculated by the fixed-sample-path method.

Step 1: Generating random numbers
Generate 2L number of random numbers under the uniform distribution on an interval $(0,1)$, and represent them as $\mathbb{w}^{(k)} = \left(w_1^{(k)}, w_2^{(k)}\right)_{k=1\ldots L}$.

Step 2: Performing Monte-Carlo simulation
Calculate the system behaviour $\mathbb{x}^{(k)} = \left(x_1^{(k)}, x_2^{(k)}\right)_{k=1\ldots L}$ from $\left(w_1^{(k)}, w_2^{(k)}\right)_{k=1\ldots L}$. Although this is quite easy for this simple example, this step might be computer-intensive for real-world problems.

Step 3: Calculating $\left.\frac{\partial \mathbb{w}}{\partial \mathbb{z}}\right|_{x=\mathrm{const}}$
For all $\left(w_1^{(k)}, w_2^{(k)}\right)_{k=1\ldots L}$, calculate

$$\begin{aligned} \left.\frac{\partial w_1}{\partial z}\right|_{x=\mathrm{const}} &= -\frac{\partial x_1}{\partial z}\Big/\frac{\partial x_1}{\partial w_1} = -\frac{w_1}{z}\,, \\ \left.\frac{\partial w_2}{\partial z}\right|_{x=\mathrm{const}} &= -\left(\frac{\partial x_2}{\partial z} + \frac{\partial x_2}{\partial w_1}\left.\frac{\partial w_1}{\partial z}\right|_{x=\mathrm{const}}\right)\Big/\frac{\partial x_2}{\partial w_2} \\ &= \frac{w_1 - 4z\, w_2 - 8z^2}{4z^2 + 2z\, w_2}\,. \end{aligned} \tag{27}$$

If an analytical calculation is impossible, we can adopt numerical approaches. For example, considering a small Δz (e.g. $\Delta z = 0.01$), find numerically $(\Delta w_1, \Delta w_2)$, which satisfies $\mathbb{x}((w_1, w_2), z) = \mathbb{x}\left((w_1 + \Delta w_1, w_2 + \Delta w_2), z + \Delta z\right)$, and approximate $\left.\frac{\partial w_j}{\partial z}\right|_{x=\mathrm{const}}$ by $\frac{\Delta w_j}{\Delta z}$, $(j = 1, 2)$.

From Eq. (27), $\left.\frac{\partial \mathbb{w}}{\partial \mathbb{z}}\right|_{x=\mathrm{const}}$ turns out to be non-zero on the boundary $\partial\Omega$, and therefore, consideration of the boundary residual R is required.

Step 4: Calculating $\frac{\partial}{\partial w_j}\left(\left.\frac{\partial w_j}{\partial z}\right|_{x=\mathrm{const}}\right)$
Calculate the derivatives of $\left.\frac{\partial w_j}{\partial z}\right|_{x=\mathrm{const}}$ with respect to w_j:

$$\frac{\partial}{\partial w_1}\left(\frac{\partial w_1}{\partial z}\bigg|_{x=\text{const}}\right) = -\frac{1}{z}\,,$$
$$\frac{\partial}{\partial w_2}\left(\frac{\partial w_2}{\partial z}\bigg|_{x=\text{const}}\right) = -\frac{w_1}{2z(2z+w_2)^2}\,. \tag{28}$$

We can also calculate them numerically by finding $\frac{\partial w_j}{\partial z}\Big|_{x=\text{const}}$ under a small variation of w_j.

Step 5: Calculating the score function $h(\mathbb{w},\mathbb{z})$
From Eq. (19), which is the case for uniformally distributed random variables, we obatin

$$h(\mathbb{w},z) = \sum_{j=1}^{2} \frac{\partial}{\partial w_j}\left(\frac{\partial w_j}{\partial z_i}\bigg|_{x=\text{const}}\right) = -1 - \frac{w_1}{2z(w_2+2z)^2}. \tag{29}$$

Numerically, calculate the sum of $\frac{\partial}{\partial w_j}\left(\frac{\partial w_j}{\partial z}\Big|_{x=\text{const}}\right)$.

Step 6: Calculating the sensitivity $A'(z)$
To derive $A'(z)$, the boundary residual R is required. R is calculated by using line 2 of Eq. (17):

$$\begin{aligned} R &= \textstyle\int_0^1 a(x)\{\, s(\mathbb{w},z)|_{w_2=1} - s(\mathbb{w},z)|_{w_2=0}\}\, dw_1 \\ &\quad + \textstyle\int_0^1 a(x)\{\, s(\mathbb{w},z)|_{w_1=1} - s(\mathbb{w},z)|_{w_1=0}\}\, dw_2 \\ &= \frac{-1}{192\, z^4\, (2z+1)}\{16\, z^2\, (6z+1)(48z^3+32z^2+7z+1) \\ &\quad +3(128z^4+64z^3+1)\sin(4z) - 12z\cos(4z)\}\,. \end{aligned} \tag{30}$$

Calculation using line 3 of Eq. (17) leads to the same result. Once R is obtained, we can calculate $A'(z)$ using Eq. (8):

$$\begin{aligned} A'(z) &= \int a(\mathbb{z})\, h(\mathbb{w},z)\, g(\mathbb{w})\, d\mathbb{w} - R \\ &= \iint_{\Omega} \{\sin(4z\, w_1) + (z\, w_2)^2 + w_1\} \\ &\qquad \left\{-1 - \frac{w_1}{2\,(w_2+2)^2}\right\} dw_1\, dw_2 - R \\ &= \frac{4z\sin(4z) + \cos(4z) + 8z^2(4z+1) - 1}{4z^2}\,, \end{aligned} \tag{31}$$

which is the same result of direct calculation Eq. (26), as expected. Numerically, apply the Monte-Carlo LRM method with the score function $s(\mathbb{w},\mathbb{z})$:

$$A'(z) \cong \frac{1}{L}\sum_{k=1}^{L} a\big(\mathbb{x}^{(k)}\big)\, h\big(\mathbb{w}^{(k)},z\big) - R\,. \tag{32}$$

3.2 Risk Measures (Greeks) in Finance

Currently, financial engineering is one of the most active fields of investigation that uses the Monte-Carlo method, and option pricing and designing hedge strategies are especially important applications.

Let us calculate some typical risk measures (Greeks), Delta Δ, Vega ν, and Rho ρ for an Asian European call option by using LRM with the fixed-sample-path method. We suppose the underlying asset price $X(t)$ of the option follows a geometric Brownian motion (GBM) under a risk-neutral probability measure,

$$dX(t) = r\, X(t)\, dt + \sigma\, X(t)\, dB(t), \tag{33}$$

with the spot (initial) price $X(0) = X_0 > 0$, where r is a risk-free interest rate, σ is the volatility of the asset price, and $B(t)$ is a standard Brownian motion. Equation (33) has an explicit solution:

$$X(t) = X(0)\exp\left[\left(r - \tfrac{\sigma^2}{2}\right)t + \sigma\, B(t)\right]. \tag{34}$$

The discounted value C_A of an Asian (average-price) European call option derived from this asset with expiration date T and strike price K satisfies

$$C_A = e^{-rT}\, \mathbb{E}\left[\max\left(\tfrac{1}{T}\textstyle\int_0^T X(t)\, dt - K, 0\right)\right], \tag{35}$$

where $\mathbb{E}[\cdot]$ denotes the expectation under the risk-neutral probability measure. Dividing T into M segments, we discretize Eq. (34) to

$$X_{j+1} = X_j \exp\left[\left(r - \tfrac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}\, w_{j+1}\right], \tag{36}$$

where $X_j = X(j\Delta t)$ is the system behaviour, where $j = 1, \dots, M$ and $\Delta t = T/M$, and $\{w_j\}_{j=1\dots M}$ are independent standard normal random variables. Then, approximating continuous-time integral of Eq. (35) by discrete-time summation leads to

$$\begin{aligned} C_A &= e^{-rT}\, \mathbb{E}\left[\max\left(\frac{1}{M}\sum_{j=1}^{M} X_j - K, 0\right)\right] \\ &= e^{-rT}\, \mathbb{E}[a(X)] = e^{-rT}\, A(X_0,\sigma,\rho)\,, \end{aligned} \tag{37}$$

where we define the behaviour evaluation function as $a(X) = \max\left(\frac{1}{M}\sum_{j=1}^{M} X_j - K, 0\right)$, which equals the payoff function of the option, and its expectation as $A(X_0,\sigma,\rho) = \mathbb{E}[a(X)]$.

On the basis of the above preparations, let us calculate three typical risk measures (Greeks), Delta Δ, Vega ν, and Rho ρ, defined as

$$\begin{aligned} \Delta &= \frac{\partial C_A}{\partial X_0} = e^{-rT}\frac{\partial A}{\partial X_0} \\ \nu &= \frac{\partial C_A}{\partial \sigma} = e^{-rT}\frac{\partial A}{\partial \sigma} \\ \rho &= \frac{\partial C_A}{\partial r} = e^{-rT}\frac{\partial A}{\partial r} - T\, C_A\,. \end{aligned} \tag{38}$$

We note that since the distribution range Ω of $\{W_j\}_{j=1\dots M}$ covers the whole $\mathbb{R}^M$, the boundary residual R_i equals ZERO.

3.2.1 Delta $\Delta = \partial C_A/\partial X_0$

Delta Δ, which represents the sensitivity of option value C_A with respect to the spot price of the underlying asset, called "Delta Δ ", is the most fundamental Greek in option trading. Considering the small variation of random variables $\{w_j\}_{j=1\ldots M}$ that cancel out a given small variation of the spot price X_0, we easily obtain the fixed-sample-path derivative

$$\begin{cases} \left.\frac{\partial w_1}{\partial X_0}\right|_{x=\mathrm{const}} = -\frac{\partial X_1}{\partial X_0} \Big/ \frac{\partial X_1}{\partial w_1} = -\frac{1}{\sigma\sqrt{\Delta t}\,X_0}, \\ \left.\frac{\partial w_j}{\partial X_0}\right|_{x=\mathrm{const}} = 0 \ \ for\ \ j \geq 2\,. \end{cases} \tag{39}$$

The score function can be calculated by Eq. (20):

$$h_{X_0} = \frac{w_1}{\sigma\sqrt{\Delta t}\,X_0}. \tag{40}$$

Therefore,

$$\begin{aligned} \Delta &= e^{-rT}\frac{\partial A}{\partial X_0} = e^{-rT}\,\mathbb{E}\left[a(\mathbb{X})\,h_{X_0}\right] \\ &= e^{-rT}\,\mathbb{E}\left[a(X)\,\frac{W_1}{\sigma\sqrt{\Delta t}\,X_0}\right] \end{aligned} \tag{41}$$

holds. We can estimate Δ by Monte-Carlo expectation (LRM) by using Eq. (41) with a large number of sample paths generated by Eq. (35).

3.2.2 Vega $\nu = \partial C_A/\partial\sigma$

The sensitivity of option value C_A with respect to the volatility σ of the asset price is called "Vega ν". The fixed-sample-path derivative with respect to σ is

$$\left.\frac{\partial w_j}{\partial\sigma}\right|_{x=\mathrm{const}} = -\frac{\partial X_j}{\partial\sigma}\Big/\frac{\partial X_j}{\partial w_j} = \sqrt{\Delta t} - \frac{w_j}{\sigma}, \tag{42}$$

and the score function is

$$h_\sigma = \sum_{j=1}^{M}\left[\frac{(w_j)^2-1}{\sigma} - \sqrt{\Delta t}\,w_j\right] = \frac{\overline{(w_j)^2} - \sigma\sqrt{\Delta t}\,\overline{w_j} - M^2}{M\,\sigma}, \tag{43}$$

where we use the notations $\overline{w_J} = \frac{1}{M}\sum_{j=1}^{M} w_j$ and $\overline{(w_J)^2} = \frac{1}{M}\sum_{j=1}^{M}(w_j)^2$. Therefore, we obtain LRM estimator

$$\begin{aligned} \nu &= e^{-rT}\frac{\partial A}{\partial\sigma} = e^{-rT}\,\mathbb{E}[a(X)\,h_\sigma] \\ &= e^{-rT}\,\mathbb{E}\left[a(X)\,\frac{\overline{(W_J)^2} - \sigma\sqrt{\Delta t}\,\overline{W_J} - M^2}{M\,\sigma}\right]. \end{aligned} \tag{44}$$

3.2.3 Rho $\rho = \partial C_A/\partial r$

The sensitivity of option value C_A with respect to the risk-free interest rate r is called "Rho ρ ". Considering the small variation of random variables $\{w_j\}_{j=1\ldots M}$ that cancel out a given small variation of the risk-free interrest r , the fixed-sample-path derivative is

$$\left.\frac{\partial w_j}{\partial r}\right|_{x=\mathrm{const}} = -\frac{\partial X_j}{\partial r}\Big/\frac{\partial X_j}{\partial w_j} = -\frac{\sqrt{\Delta t}}{\sigma} \tag{45}$$

The score function calculated by Eq. (20) is

$$h_r = \sum_{j=1}^{M}\frac{\sqrt{\Delta t}\,w_j}{\sigma} = \frac{\sqrt{\Delta t}}{M\,\sigma}\,\overline{w_J}\,. \tag{46}$$

Therefore, we obtain the LRM estimator

$$\begin{aligned} \rho &= e^{-rT}\frac{\partial A}{\partial r} - T\,C_A = e^{-rT}\,\mathbb{E}[a(\mathbb{X})\,(h_r - T)] \\ &= e^{-rT}\,\mathbb{E}\left[\,a(X)\left(\frac{\sqrt{\Delta t}}{M\,\sigma}\,\overline{W_J} - T\right)\right]. \end{aligned} \tag{47}$$

As might be expected, the score functions and LRM estimators of Delta Δ, Vega ν, and Rho ρ derived from the fixed-sample-path method in this section are the same as those derived from the conventional method by differentiating the probability density function (Glasserman, 2003, Broadie and Glasserman, 1996). It is noteworthy that the conventional method requires explicit knowledge of the relevant probability density function, whereas the fixed-sample-path method requires the knowledge of the time evolution of individual sample paths x only.

4 ANALYSIS OF FINANCIAL FLOW-OF-FUNDS NETWORK

The calculation examples of Section 3 were aimed at pretty simple systems. In this section, we address a network model of the financial flow of funds among companies as an example of the relatively complicated system that shows the effectiveness of the fixed-sample-path method.

4.1 Outline of the Problem

Let us consider a network of the financial flow of funds among 25 companies, labeled 1 to 25, as shown in Figure 1. While a network consisting of 25 companies is not too complicated to understand and discuss the results, it is fairly complicated to perform sensitivity analysis with the conventional LRM method. In Figure 1, the nodes represent each company, where the numbers written in the nodes represent the company's label. The edges represent the existence of the financial flow of funds along the edge directions. For simplicity, we suppose that the average amounts of fund transfers per unit period equals one for all edges. We suppose, in addition, that the assets of each company increase or decrease by an average amount per unit period denoted by parenthetical numbers beside each node, whereas the assets of the companies of which corresponding nodes have no parenthetical numbers do not change. This increase or decrease in assets represents the fund transfers from/to companies other than those of the 25 companies depicted in Figure 1. As a result, the average net incomes and outgoings per unit period of each of the 25 companies are balanced.

We suppose the actual amounts of fund transfers through the edges to be random variables distributed around the above average amounts. The assets of each company increase or decrease depending on the variation of the difference between incomes and outgoings. As a result, there is the possibility for "company bankruptcy", i.e., the assets of a certain company go negative at a certain time. Here, we suppose that companies in bankruptcy and the edges (funds transfer) related to them cease to exist. If company 1 in Figure 1, for example, goes bankrupt at time t, we delete four edges: from Co. 1 to Co. 3, Co. 1 to Co. 20, Co. 17 to Co. 1, and Co. 18 to Co. 1. As a consequence, companies 3 and 20 become increasingly likely to go bankrupt because of an unfavourable balance without fund transfers from company 1, whereas companies 17 and 18 become less likely to go bankrupt because of a favourable balance. Bankruptcy of a company has an effect on the bankrupt probabilities of the other companies through the connection structure of the network in this way.

Now, we are interested in the relationship between the flow of funds of the edges and the bankrupt probabilities of the companies. If the average flows of each edge slightly change from 1, what happens in the bankrupt probability of company 1 or the average bankrupt probability of all 25 companies? Conversely, which edge is the most effective at reducing the bankrupt probability of company 1 if we change the average flow of funds? The edges linked directly from/to company 1 might naturally have a large influence, but is there a possibility that edges located away from company 1 have a large influence on its bankrupt probability by network effect? Given this awareness of the problems, the aim of this section is to estimate the sensitivities of the bankrupt probabilities of each company and the sensitivity of the average bankrupt probability of the all companies with respect to the average flow of funds of each edge by Monte-Carlo simulation by using LRM with the fixed-sample-path principle.

4.2 Formulation

Let us consider a network of the financial flow of funds among 25 companies, shown in Figure 1. We call the "outside" of the network as "company 0" for notational convenience, i.e., the fund transfers from/to companies outside the network (denoted by parenthetical numbers beside each node) are considered to be the fund transfers from/to company 0. Let $X_i(t)$ denote the total assets of company i (where $i = 1 \ldots 25$) at time t. We suppose the initial assets $X_i(0) = 25$ for all 25 companies. The existence function of company i is defined as

$$S_i(t) = \begin{cases} 1, & \text{if } X_i(t) \geq 0 \\ 0, & \text{otherwise} \end{cases}, \tag{48}$$

i.e., $S_i(t)$ equals 1 if company i exists at time t, and $S_i(t)$ equals 0 if company i has been bankrupt. We define $S_0(t) = 1$ for all t for notational simplicity. Let $F_{ij}(t)$ denote the amount of the transfer of funds from company i to company j at time t. $F_{ij}(t)$ are random variables with mean $\mu_{ij} = 1$ for i, j (where $i = 1, \ldots, 25$ and $j = 1, \ldots, 25$) for which there exists an edge between company i and j, while μ_{ij} equals zero for i, j for which there exists no edge between them. In addition, $F_{0j}(t)$ and $F_{i0}(t)$, which denote the transfer of funds from/to the outside of the network, are random variables with mean μ_{0j}, μ_{i0}=1–3, shown in parentheses in the figure. Here, we suppose $F_{ij}(t)$ to be under log-normal distribution with mean μ_{ij} and variance $\sqrt{\mu_{ij}}$. The assets $X_i(t)$ of company (where $i = 1 \ldots 25$) satisfy the relation

$$\begin{aligned} X_i(t+1) - X_i(t) \\ = \sum_{j=0}^{25} F_{ji}(t)\, S_j(t)\, S_i(t) - \sum_{j=0}^{25} F_{ij}(t)\, S_i(t) S_j(t) \end{aligned} \tag{49}$$

On the basis of the above premises, let us estimate the existence probabilities $\mathcal{S}_i = \mathbb{E}[S_i(T)]$ of each company at $T = 100$ and the average existence probability $\overline{\mathcal{S}_\imath} = \mathbb{E}\left[\frac{1}{25}\sum_{i=1}^{25} S_i(T)\right]$ of all 25 companies by Monte-Carlo simulation. In addition, we estimate $\partial \mathcal{S}_i / \partial \mu_{ij}$ and $\partial \overline{\mathcal{S}_\imath} / \partial \mu_{ij}$, i.e., the sensitivity of $\mathcal{S}_i$ and $\overline{\mathcal{S}_\imath}$ with respect to the average flow of funds of each edge, by using the fixed-sample-path method. There exist 86 numbers of μ_{ij}, which are non-zero, that is, 71 edges plus 15 parenthetical numbers. We can estimate $\partial \mathcal{S}_i/\partial\mu_{ij}$ and $\partial\,\overline{\mathcal{S}_i}/\partial\mu_{ij}$ for all 86 μ_{ij} simultaneously.

4.3 Derivation of Score Functions

To estimate $\partial \mathcal{S}_i/\partial\mu_{ij}$ and $\partial\,\overline{\mathcal{S}_i}/\partial\mu_{ij}$, the score functions $h_{(\mu_{ij})}$ are required. The log of a random variable under log-normal distribution with mean μ and variance σ is under normal distribution with mean m and variance s:

$$\begin{cases} m = \log\left(\dfrac{\mu^2}{\sqrt{\mu^2+\sigma^2}}\right) \\ s = \sqrt{\log\left(1+\dfrac{\sigma^2}{\mu^2}\right)}. \end{cases} \tag{50}$$

Therefore, $F_{ij}(t)$, which is under log-normal distribution with mean μ_{ij} and variance $\sqrt{\mu_{ij}}$, can be written as

$$\begin{aligned} F_{ij}(t) &= \exp(m_{ij} + s_{ij}\, w_{ij}^t) \\ &= \frac{(\mu_{ij})^2 \sqrt{\log(1+1/\mu_{ij}}\ \exp(\sqrt{\log(1+1/\mu_{ij}}\ w_{ij}^t)}{\sqrt{\mu_{ij}\,(1+\mu_{ij})}} \end{aligned} \tag{51}$$

where w_{ij}^t is a random variable with the standard normal distribution.

Let us apply the fixed-sample-path method. Considering the relationship between a small variation of w_{ij}^t and a small variation of μ_{ij} under the condition of keeping $F_{ij}(t)$ fixed satisfies

$$\begin{aligned} \left.\frac{\partial w_{ij}^t}{\partial \mu_{ij}}\right|_{F_{ij}(t)=\mathrm{const}} &= -\frac{dF_{ij}(t)}{d\mu_{ij}} \Big/ \frac{\partial F_{ij}(t)}{\partial w_{ij}^t} \\ &= \frac{w_{ij}^t - (3+2\,\mu_{ij})\sqrt{\log\left(1+\frac{1}{\mu_{ij}}\right)}}{2\,\mu_{ij}\,(1+\mu_{ij})\log\left(1+\frac{1}{\mu_{ij}}\right)} \end{aligned} \tag{52}$$

and the fact that the system behaviour $X_i(t)$ stays fixed if and only if all fund flows $F_{ij}(t)\,S_i(t)\,S_j(t)$ are fixed, we obtain the fixed-sample-path derivative

$$\left.\frac{\partial w_{ij}^t}{\partial \mu_{ij}}\right|_{x=\mathrm{const}} = \begin{cases} \dfrac{w_{ij}^t - (3+2\,\mu_{ij})\sqrt{\log\left(1+\frac{1}{\mu_{ij}}\right)}}{2\,\mu_{ij}\,(1+\mu_{ij})\log\left(1+\frac{1}{\mu_{ij}}\right)}, & if\ \ S_i(t)\,S_i(t)=1 \\ 0, & otherwise. \end{cases} \tag{53}$$

Therefore, from Eq. (20), he score function $h_{(\mu_{ij})}$ with respect to μ_{ij} is

$$\begin{aligned} h_{(\mu_{ij})} &= \sum_{t=1}^{\min(\tau_i,\tau_j)} \left\{ \frac{\partial}{\partial w_{ij}^t}\left(\left.\frac{\partial w_{ij}^t}{\partial \mu_{ij}}\right|_{x=\mathrm{const}} \right) - w_{ij}^t \left(\left.\frac{\partial w_{ij}^t}{\partial \mu_{ij}}\right|_{x=\mathrm{const}} \right) \right\} \\ &= \sum_{t=1}^{\min(\tau_i,\tau_j)} \frac{1-(w_{ij}^t)^2 + w_{ij}^t\,(3+2\,\mu_{ij})\sqrt{\log(1+1/\mu_{ij})}}{2\,\mu_{ij}\,(1+\mu_{ij})\log(1+1/\mu_{ij})} \end{aligned} \tag{54}$$

where τ_i is the last time that company i exists:

$$\tau_i = \mathrm{argmax}_{t\le T}[\,S_i(t)=1\,]\ . \tag{55}$$

4.4 Simulation Result

We performed a Monte-Carlo simulation with two-million sample paths and estimated

$$\begin{aligned} \mathcal{S}_i &= \mathbb{E}[S_i(100)] \cong \frac{1}{L}\sum_{k=1}^{L} S_i^{(k)}(100) \\ \overline{\mathcal{S}_i} &= \mathbb{E}\left[\frac{1}{25}\sum_{i=1}^{25} S_i(100)\right] \cong \frac{1}{L}\sum_{k=1}^{L}\left[\frac{1}{25}\sum_{i=1}^{25} S_i(100)\right] \\ \frac{\partial \mathcal{S}_i}{\partial \mu_{ij}} &= \mathbb{E}\left[S_i(100)\,h_{(\mu_{ij})}\right] \cong \frac{1}{L}\sum_{k=1}^{L} S_i^{(k)}(100)\,h_{(\mu_{ij})}^{(k)} \\ \frac{\partial \overline{\mathcal{S}_i}}{\partial \mu_{ij}} &= \mathbb{E}\left[\tfrac{1}{25}\textstyle\sum_{i=1}^{25} S_i(100)\,h_{(\mu_{ij})}\right] \\ &\cong \tfrac{1}{L}\textstyle\sum_{k=1}^{L}\left[\tfrac{1}{25}\textstyle\sum_{i=1}^{25} S_i^{(k)}(100)\,h_{(\mu_{ij})}^{(k)}\right]\ . \end{aligned} \tag{56}$$

Figure 2 shows an over-drawn time series of 200 typical Monte-Carlo sample paths of $\frac{1}{25}\sum_{i=1}^{25} S_i(t)$, the average existence probability of the 25 companies. Figures 3 - 5 show the convergence of the estimated values: Fig. 3 for $\mathcal{S}_i$ and $\overline{\mathcal{S}_i}$, Fig. 4 for $\partial\,\overline{\mathcal{S}_i}/\partial\mu_{ij}$, and Fig. 5 for $\partial \mathcal{S}_1/\partial\mu_{ij}$. All of the estimated values are converged. As is known, the convergence speeds of the sensitivities when using the LRM method are slower than those of the expectations themselves (Glasserman, 2003).

Table 1 shows the estimated values of $\overline{\mathcal{S}_i}$ and $\mathcal{S}_i$ and their sensitivities $\partial\,\overline{\mathcal{S}_i}/\partial\mu_{ij}$ and $\partial \mathcal{S}_1/\partial\mu_{ij}$. The leftmost col umn of the table shows the estimated value of $\overline{\mathcal{S}_i}$ (the average existence probability of the 25 companies) and the 25 estimated values of $\mathcal{S}_i$ (the existence probabilities of company i). The right ten columns of the table show the sensitivities (differential coefficients) of $\overline{\mathcal{S}_i}$ and $\mathcal{S}_i$ with respect to the average funds flow μ_{ij} of edges. Due to limited space, the sensitivities with respect to only ten edges, arranged in descending order of their absolute values, are shown respectively, where the upper rows identify the edges, and the bottom rows show the estimated values of the differential coefficients.

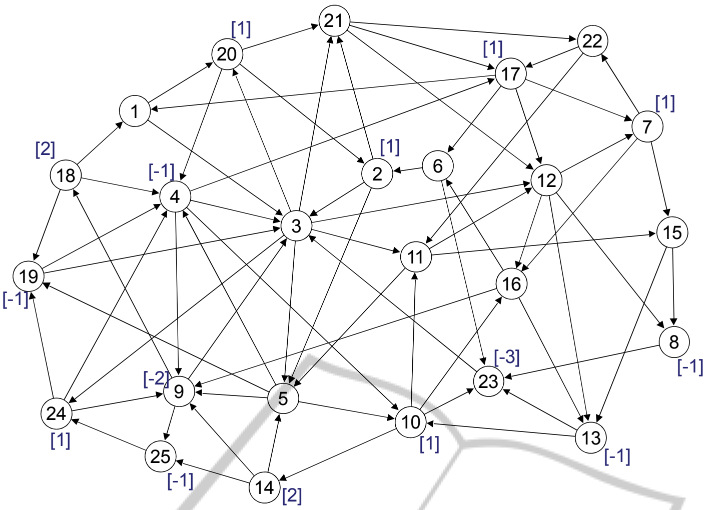

Figure 1: Financial flow-of-funds network with 25 companies.

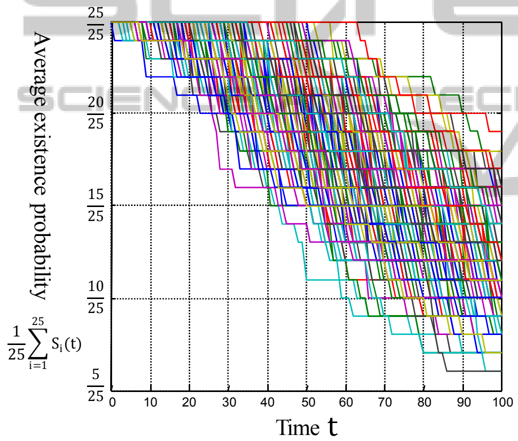

Figure 2: Over-drawn time series of the average existence probability by Monte-Carlo simulation.

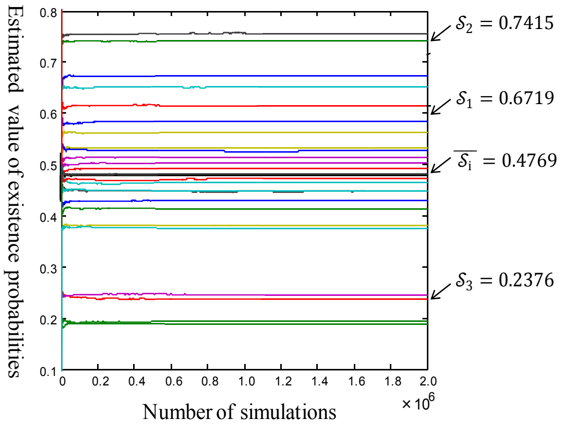

Figure 3: Estimated value of existence probabilities vs number of simulations.

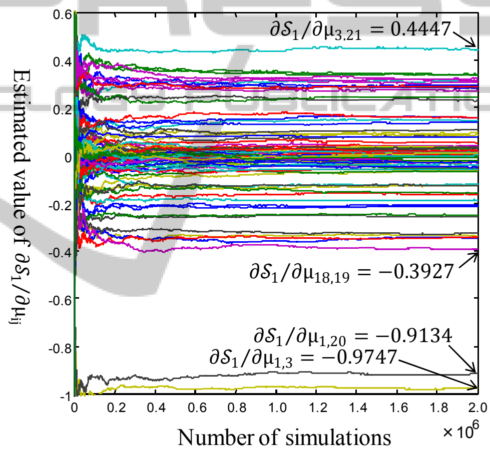

Figure 4: Estimated value of $\partial\overline{\mathcal{S}_i}/\partial\mu_{ij}$ vs number of simulations.

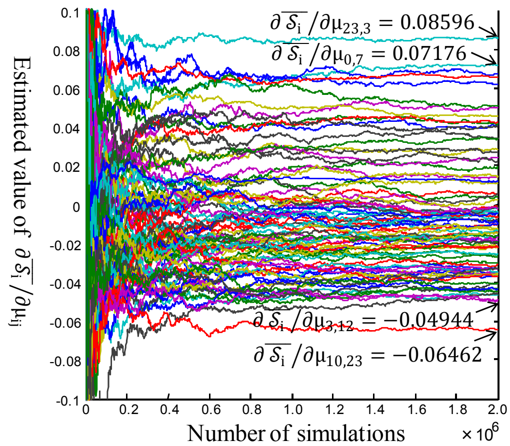

Figure 5: Estimated value of $\partial\mathcal{S}_1/\partial\mu_{ij}$ vs number of simulations.

Table 1: the estimated values of $\overline{\mathcal{S}_i}$ and $\mathcal{S}_i$ and their sensitivities $\partial\overline{\mathcal{S}_i}/\partial\mu_{ij}$ and $\partial\mathcal{S}_1/\partial\mu_{ij}$.

	Existence Probability	The estimated sensitivities with respect to each edge μ_{ij}									
		#1	#2	#3	#4	#5	#6	#7	#8	#9	#10
$\overline{\mathcal{S}_i}$	0.4769	23→3	0→7	0→10	0→18	10→23	0→17	0→20	0→2	3→12	3→24
		0.08596	0.07176	0.06742	0.06573	-0.06462	0.06146	0.05095	0.04989	-0.04944	-0.04906
$\mathcal{S}_1$	0.6719	1→3	1→20	3→21	18→19	17→6	3→11	17→7	18→4	18→1	17→12
		-0.9747	-0.9134	0.4447	-0.3927	-0.3461	0.3424	-0.3414	-0.3368	0.3345	-0.3238
$\mathcal{S}_2$	0.7415	2→3	2→5	2→21	0→2	3→20	20→21	6→2	20→2	6→23	20→4
		-0.8100	-0.7856	-0.7707	0.6345	0.4148	-0.3370	0.3240	0.3066	-0.2995	-0.2838
$\mathcal{S}_3$	0.2376	3→21	3→11	3→20	3→12	3→24	20→4	3→5	2→21	1→20	20→2
		-0.5485	-0.5407	-0.5334	-0.4134	-0.3956	0.3719	-0.3607	-0.2965	-0.2859	0.2798
$\mathcal{S}_4$	0.4635	4→10	4→9	4→17	4→0	4→3	9→18	24→9	5→9	5→10	19→4
		-0.7510	-0.7464	-0.7339	-0.6120	-0.4982	0.4884	-0.4323	-0.3803	-0.3614	0.3497
$\mathcal{S}_5$	0.5024	5→19	5→4	5→9	5→10	4→3	19→3	9→3	3→24	3→12	3→21
		-0.7143	-0.7053	-0.6733	-0.6124	0.4746	0.4496	0.4455	-0.4150	-0.3834	-0.3569
$\mathcal{S}_6$	0.5322	6→23	6→2	16→13	23→3	17→12	16→9	17→1	23→0	0→17	12→16
		-1.0721	-1.0054	-0.4793	0.4554	-0.4080	-0.4065	-0.4058	0.3542	0.3474	0.3396
$\mathcal{S}_7$	0.4482	7→16	7→15	7→22	0→7	22→17	12→16	17→1	17→6	16→9	0→17
		-1.0488	-1.0205	-0.9594	0.7789	0.5961	-0.4402	-0.4348	-0.3906	0.3893	0.3809
$\mathcal{S}_8$	0.4284	8→23	8→0	23→3	15→13	12→13	23→0	12→16	15→8	3→5	12→8
		-1.0896	-0.8440	0.6284	-0.5632	-0.5147	0.4214	-0.4110	0.3514	-0.2901	0.2867
$\mathcal{S}_9$	0.1940	9→25	9→18	9→0	25→24	9→3	14→25	18→4	24→19	14→9	5→19
		-0.5920	-0.5796	-0.4650	0.3994	-0.3581	-0.3199	0.3104	-0.2640	0.2539	-0.2515
$\mathcal{S}_{10}$	0.4711	10→23	10→16	10→11	10→14	0→10	11→5	13→23	16→13	14→5	5→9
		-0.8638	-0.8049	-0.7756	-0.7606	0.6648	0.4750	-0.4707	0.4213	0.4069	-0.3430
$\mathcal{S}_{11}$	0.4483	11→15	11→12	11→5	5→10	12→13	22→17	3→12	15→13	10→14	3→11
		-0.9037	-0.8759	-0.8294	0.4712	0.4209	-0.3898	-0.3650	0.3629	-0.3574	0.3462
$\mathcal{S}_{12}$	0.2472	12→7	12→16	12→8	12→13	7→22	17→7	16→9	0→7	8→23	3→5
		-0.6790	-0.5719	-0.5492	-0.5186	0.3617	-0.3588	0.2915	-0.2793	0.2644	-0.2593
$\mathcal{S}_{13}$	0.3799	13→10	13→23	13→0	10→16	10→11	12→8	23→3	15→8	16→6	3→5
		-0.8373	-0.8043	-0.6842	0.5073	0.4574	-0.4472	0.4331	-0.4145	-0.3261	-0.3106
$\mathcal{S}_{14}$	0.7548	14→9	14→5	14→25	0→14	5→10	10→16	10→11	10→23	0→10	9→18
		-0.8365	-0.7869	-0.7464	0.6162	0.3702	-0.3328	-0.3267	-0.3026	0.2715	0.2418
$\mathcal{S}_{15}$	0.5260	15→8	15→13	12→7	11→12	7→16	13→10	8→23	13→23	0→7	7→22
		-1.1379	-1.1117	0.5564	-0.5426	-0.5284	0.4491	0.4376	0.4156	0.4036	-0.3907
$\mathcal{S}_{16}$	0.4141	16→6	16→13	16→9	13→10	12→13	10→14	9→3	10→11	9→18	0→10
		-0.8364	-0.8072	-0.8043	0.5246	-0.4697	-0.4008	0.3846	-0.3574	0.3417	0.3259
$\mathcal{S}_{17}$	0.4926	17→12	17→6	17→1	17→7	0→17	21→12	12→8	22→17	12→13	7→22
		-0.9313	-0.8432	-0.8264	-0.7960	0.6776	-0.5995	0.3746	0.3744	0.3742	0.3387
$\mathcal{S}_{18}$	0.6510	18→19	18→4	18→1	0→18	4→9	9→25	9→0	5→9	24→9	9→3
		-0.9755	-0.9174	-0.9007	0.7450	0.5324	-0.4179	-0.3943	0.3700	0.3544	-0.3315
$\mathcal{S}_{19}$	0.5141	19→4	19→0	19→3	18→4	24→4	5→4	18→19	24→19	4→17	5→19
		-1.1736	-0.8793	-0.7153	-0.6266	-0.6078	-0.5807	0.4290	0.4074	0.4007	0.3902
$\mathcal{S}_{20}$	0.5610	20→21	20→2	20→4	0→20	21→17	3→21	2→3	4→3	4→17	1→20
		-0.9482	-0.9310	-0.8908	0.7290	0.4338	-0.4250	0.3988	0.3926	0.3468	0.3327
$\mathcal{S}_{21}$	0.4799	21→17	21→22	21→12	3→12	17→1	2→21	17→6	12→16	20→4	12→13
		-0.9103	-0.8803	-0.8600	-0.4702	0.4002	0.3791	0.3728	0.3541	-0.3444	0.3119
$\mathcal{S}_{22}$	0.5839	22→11	22→17	17→7	21→17	7→16	7→15	11→12	0→7	17→12	11→5
		-1.0411	-0.9813	0.6397	-0.5667	-0.5137	-0.4763	0.4563	0.4280	0.3974	0.3369
$\mathcal{S}_{23}$	0.1888	23→0	23→3	3→12	6→2	13→0	10→14	8→0	6→23	0→10	10→23
		-0.5486	-0.4449	0.2889	-0.2859	-0.2788	-0.2577	-0.2489	0.2269	0.2241	0.2192
$\mathcal{S}_{24}$	0.6154	24→19	24→4	0→24	24→9	19→3	4→3	3→5	3→20	3→11	3→21
		-0.8424	-0.7966	0.6685	-0.6268	0.5091	0.5016	-0.4815	-0.4246	-0.4240	-0.4064
$\mathcal{S}_{25}$	0.3755	25→24	25→0	24→9	9→3	9→18	9→0	24→4	0→14	4→3	14→25
		-0.9278	-0.7619	0.5972	-0.5229	-0.4131	-0.3888	0.3871	0.3564	-0.3078	0.2978

From Table 1, for example, edge 23→3 (the edge from company 23 to company 3) turned out to have the largest sensitivity of 0.08596 to $\overline{\mathcal{S}_i}$. Edge 0→7 (the flow of funds from the outside of the network to company 7) and edge 0→10 (from the outside to company 10) also had large sensitivities to $\overline{\mathcal{S}_i}$. It is interesting that edge 23→3, which is an inner flow of the network, had larger sensitivity to the average existence probability $\overline{\mathcal{S}_i}$ than did the inward flows from the outside of the network, which increased the total assets within the network. This would be explained by the fact that company 23 has four inward edges, while it has only one outward edge 23→3. An increasing flow of funds for 23→3, which clearly had an adverse effect on the survival of company 23, might be desirable for the survival of the many other companies in the network.

We turn attention to the existence probability $\mathcal{S}_1$ of company 1. The top three edges having a large effect on $\mathcal{S}_1$ were edge 1→3, edge 1→20, and edge 3→21 in descending order of the (absolute value of) sensitivities. We are convinced that edge 1→3 and edge 1→20, which are directly outward from node 1, had large and negative sensitivities to $\mathcal{S}_1$. It is interesting that edge 3→21, which does not link to company 1 directly, had the third largest sensitivity. This would be explained if we note that edge 3→21 had the largest (and negative) effect on the survival of company 3 and edge 1→3 the largest (and negative) effect on the survival of company 1.

As seen above, we can estimate the sensitivity of $\overline{\mathcal{S}_i}$ and $\mathcal{S}_i$ with respect to all 86 numbers of μ_{ij} by Monte-Carlo simulation by using the LRM method with the score functions derived by using the fixed-sample-path principle. Although this example network is pretty small, the LRM method with fixed-sample-path principle can be applicable and practical for much more complicated systems with numerous parameters, such as for systematic risk analysis of complicated financial networks, traffic flow on a complicated roadway network, and emerging "big-data" analysis.

5 CONCLUSION

In this study, a fixed-sample-path method was proposed, which derives the score function of LRM not via the pdf $f(x, \mathbb{z})$. The key idea is to consider the fixed-sample-path derivative of the random variables $\mathbb{w}$ with respect to the parameter z_i under the condition of fixing the sample path x. The boundary residual R_i, which represents the correction associated with the change of the distribution range of the random variables in LRM, was also derived. Some examples including the estimation of risk measures (Greeks) of option and financial flow-of-funds networks showed the effectiveness of the fixed-sample-path method.

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