Iterative Learning Control for Linear Time-Varying Systems in the Presence of Iteration-Varying Disturbance

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Keywords: Iterative Learning Control, Linear Time-Varying Systems, 2D Roesser Model.

Abstract: This paper presents an innovative Iterative Learning Control (ILC) strategy for Linear Time-Varying (LTV) systems subject to uncertainties. In a real-world environment, implementing ILC causes the uncertainties to vary concerning both time and iteration. To address this challenge, we introduce a metric to quantify the impact of the uncertainties on the tracking error's variation. First, an equivalent 2D Roesser model is established for the uncertain ILC system. It has uncertain parameters and is subject to an external disturbance caused by the time-varying model uncertainties of the original system. Then, a Linear Matrix Inequality (LMI) condition is proposed to design the ILC law to provide an upper metric bound. The strategy aims to lower this bound, thereby reducing the impact of uncertainties on the system. Finally, preliminary numerical simulation verifies the effectiveness and robustness of the proposed strategy.

1 INTRODUCTION

Iterative Learning Control (ILC) is a control method for repetitive processes that continuously adjusts the current control signal by learning from historical operations (Bristow et al., 2006; Ahn et al., 2007; Wang et al., 2009; Owens and Hätönen, 2005; Lee and Lee, 2007). Specifically, based on the errors of previous iterations, it brings the system output closer to the expected trajectory in subsequent iterations. ILC has become an important subject of academic research since Arimoto et al. first proposed the concept of it (Arimoto et al., 1984). The self-learning character gives ILC a unique advantage in applications that perform repetitive tasks, such as robotics (Zhao et al., 2015), precision manufacturing (Hoelzle and Barton, 2014), aerospace (Yao, 2021), and power systems (Zanchetta et al., 2013). Disturbances caused by unwanted forces, torques, or environmental changes are common in practice. The ILC strategy can resist repeatable disturbance well, but iteration-varying disturbance will seriously affect the system's performance (Merry et al., 2005; Norrlöf and Gunnarsson, 2001). Therefore, suppressing the effect of iteration-varying disturbance becomes an important topic in the field of ILC.

Recently, some innovative methods have been proposed to cope with iteration-varying disturbance. Chin et al. presented a control framework that combines real-time feedback control and ILC to handle real-time disturbance in repetitive processes more efficiently (Chin et al., 2004). This combination is designed to separate the performance of the ILC from the effects of real-time disturbance, thereby improving the effectiveness of the control strategy. Maeda et al. gave a control structure that combines iterative learning control and disturbance observer (Maeda et al., 2015). The disturbance of the previous iteration is used as a partial preview of the next disturbance, thus effectively resisting near-repetitive disturbances. Sun et al. proposed a composite control scheme combining a P-type ILC scheme with extended state observer (Sun et al., 2014). Among them, the observer is used for disturbance estimation to improve the performance of systems with iteration-varying disturbance. The disturbances can be attenuated from the system output by properly selecting the compensation gain. These studies investigate the challenges of non-repetitive disturbance to ILC and highlight ongoing research efforts to address these issues. However, the above method may have the shortcomings of slow response or constrained disturbance,

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Dou, Y., Su, L. and Prempain, E.
Iterative Learning Control for Linear Time-Varying Systems in the Presence of Iteration-Varying Disturbance.
DOI: 10.5220/0012908100003822

In *Proceedings of the 21st International Conference on Informatics in Control, Automation and Robotics (ICINCO 2024) - Volume 1*, pages 645-650
ISBN: 978-989-758-717-7; ISSN: 2184-2809

so the implementation might have some difficulties.

This paper considers another ILC strategy for Linear Time-Varying (LTV) systems. Our objective is to design a suitable ILC law to mitigate the effect of the varying uncertainty on the tracking error. The strategy is based on the 2D Roesser model, which can accurately describe the bidirectional information flow and integrate the disturbance observer into the ILC process. This uncertain model is subject to an external disturbance caused by the time-varying model uncertainty of the original linear system. A similar ILC design in a 2D setting can be found in the papers (Shi et al., 2005a; Shi et al., 2005b). Then, we introduce a metric that measures the impact of the induced iteration-varying disturbance on the system. A Linear Matrix Inequality (LMI) condition is proposed to design the ILC law to provide an upper bound of the metric using the Schur complement and the S-procedure (Scherer and Weiland, 2000; Zhang, 2006; Pólik and Terlaky, 2007). Compared to the above-mentioned research, our approach adopts a simpler framework and, therefore, is easier to apply. Preliminary numerical simulations validate the effectiveness and robustness of the strategy, suggesting its potential applicability in practical scenarios.

This paper uses the following notations: $\mathbb{R}^n$ represents an n-dimensional Euclidean space. $||x||$ denotes the norm of a vector x. $\mathbb{R}^{n\times m}$ is the set of all n by m matrices with real number entries. Given $M \in \mathbb{R}^{n\times n}, M \succ 0$ indicates that M is positive definite. $M^\top$ represents the transpose of matrix M. The notations ":=" and "≡" mean "defined as" and "equivalent to", respectively. The notation "*" in matrix representations indicates that the off-diagonal block is the transpose of the corresponding lower-diagonal block.

The rest of the paper is structured as follows: In Section 2, we introduce the original system model, formulate the 2D equivalent model, and define the metric followed by the main result. Section 3 presents numerical simulation results, verifying our proposed strategy's effectiveness and robustness. Section 4 provides a brief summary and some final remarks.

2 METHODOLOGY

2.1 Discrete-Time State-Space Model

In this study, we investigate a linear time-varying system. This system captures the parameter uncertainties inherent in a process executed repetitively over multiple cycles. The model is given by the following equations (Shi et al., 2005a; Shi et al., 2005b):

$$\begin{aligned} x_k[t+1] &= (A+\delta A_k[t])x_k[t] + (B+\delta B_k[t])u_k[t], \\ y_k[t] &= Cx_k[t] \end{aligned} \tag{1}$$

where $t \in [0,1,2,\ldots,T]$ is the time index, and $k \in \mathbb{Z}^+$ is the iteration index [1]. The variables $u_k[t] \in \mathbb{R}^m$, $y_k[t] \in \mathbb{R}^l$, and $x_k[t] \in \mathbb{R}^n$ represent the input, output, and state at time t in the k-th iteration, respectively. The nominal system matrices A, B, and C define the ideal behavior of the system.

In practical applications, modeling uncertainties of the system parameters are common. In this work, we assume the modelling uncertainties of A and B can be represented by $\delta A_k[t]$ and $\delta B_k[t]$ respectively, which are defined as follows (Shi et al., 2005a; Shi et al., 2005b):

$$\begin{aligned} \delta A_k[t] &= E_1\Delta_k[t]F_1, \\ \delta B_k[t] &= E_2\Delta_k[t]F_2. \end{aligned} \tag{2}$$

The matrices E_1, E_2, F_1, and F_2 are known and of suitable dimensions, capturing the structures of the uncertain parameter perturbations. The matrix $\Delta_k[t]$ represents the unknown perturbation matrix, which is subject to the norm-bounded condition $\Delta_k^{\mathrm{T}}[t]\Delta_k[t] \leq I$ for all $t \geq 0$ and $k > 0$.

To improve the process performance over iterations, we employ the following updating law (Aarnoudse et al., 2025):

$$u_k[t] = u_{k-1}[t] + r_k[t] \tag{3}$$

where the term $r_k[t]$ signifies the learning update at time t in the k-th iteration. By default, the term $u_0[t]$ is assumed to be a zero sequence.

2.2 Formulation of 2D Roesser System

ILC is a strategy designed for repetitive tasks characterized by a two-dimensional system with time and iteration as two independent coordinates. The Roesser model, a 2D system model, is well suited for representing ILC because it can depict information flow in two directions. By employing this model, one can combine the evolution of the state variable in the domains of time and iteration. The combination aids in developing efficient learning algorithms and robust system convergence and stability analysis.

In what follows, we use $f(t,k)$ to denote $f_k[t]$, and define $d(f(t,k)) := f(t,k) - f(t,k-1)$. Applying this definition to the set of quantities $\{x,u,y,r,e,\delta A,\delta B,\Delta\}$, we can derive from equations (1) and (3) the following:

[1]The index of k can be dropped in terms of describing the linear time-varying plant. k is added to facilitate the analysis with ILC in the following.

$$\begin{aligned} d(x(t+1,k)) = & \quad (A+\delta A(t,k))d(x(t,k)) \\ & + (B+\delta B(t,k))r(t,k)+w(t,k) \end{aligned} \tag{4}$$

where

$$\begin{aligned} w(t,k) = & \quad d(\delta A(t,k))x(t,k-1) \\ & + d(\delta B(t,k))u(t,k-1). \end{aligned} \tag{5}$$

In the case of repeatable parameter perturbation, i.e., $d(\delta A(t,k)) \equiv 0$ and $d(\delta B(t,k)) \equiv 0$, the term $w(t,k)$ reduces to $w(t,k) \equiv 0$. In general, the time-varying modeling uncertainties are non-repeatable [2], and therefore $d(\delta A(t,k)) \neq 0$ and $d(\delta B(t,k)) \neq 0$. The induced term $w(t,k)$ is non-trivial and referred to as an iteration-varying disturbance in this work.

Let us denote the tracking error at time $t+1$ in the k-th iteration as $e(t+1,k)$, i.e., $e(t+1,k) := y_r(t+1) - y(t+1,k)$. It can be shown that $e(t+1,k)$ is related to $e(t+1,k-1)$ by the following equation:

$$\begin{aligned} e(t+1,k) = & -C(A+\delta A(t,k))d(x(t,k)) \\ & -C(B+\delta B(t,k))r(t,k) \\ & -Cw(t,k)+e(t+1,k-1). \end{aligned} \tag{6}$$

Note from the above equation that the variation of tracking error $d(e(t+1,k))$ is affected by the unpredictable varying perturbation $w(t,k)$. It is important to design a learning law to mitigate the effect of the perturbation on the tracking error.

Combining equations (4) and (6), we obtain the 2D model denoted as:

$$\begin{aligned} \Sigma: \begin{bmatrix} d(x(t+1,k)) \\ e(t+1,k) \end{bmatrix} = & (\bar{A}+\delta\bar{A}) \begin{bmatrix} d(x(t,k)) \\ e(t+1,k-1) \end{bmatrix} \\ & + (\bar{B}+\delta\bar{B})r(t,k)+\bar{D}w(t,k) \end{aligned} \tag{7}$$

where

$$\bar{A} = \begin{bmatrix} A & 0 \\ -CA & I \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B \\ -CB \end{bmatrix}, \quad \bar{D} = \begin{bmatrix} I \\ -C \end{bmatrix},$$

$$\begin{aligned} \delta\bar{A} &= \begin{bmatrix} \delta A(t,k) & 0 \\ -C\delta A(t,k) & 0 \end{bmatrix} \\ &= \bar{E}_1\Delta(t,k)\bar{F}_1 = \begin{bmatrix} E_1 \\ -CE_1 \end{bmatrix} \Delta(t,k) \begin{bmatrix} F_1 & 0 \end{bmatrix}, \\ \delta\bar{B} &= \begin{bmatrix} \delta B(t,k) \\ -C\delta B(t,k) \end{bmatrix} \\ &= \bar{E}_2\Delta(t,k)\bar{F}_2 = \begin{bmatrix} E_2 \\ -CE_2 \end{bmatrix} \Delta(t,k)F_2. \end{aligned} \tag{8}$$

[2] For example, $\delta A, \delta B$ may be modeling error caused by the linearization approximation, which is trajectory-dependent. The trajectory in different iterations is variant, and hence $\delta A, \delta B$ varies with respect to k.

Let us interpret $[d(x(t+1,k)), e(t+1,k)]^\top$, $r(t,k)$, and $w(t,k)$ be the state, input, and disturbance, respectively. Then, the system Σ can be viewed as a 2D Roesser model that incorporates uncertain parameter perturbation and external disturbance. This model is particularly advantageous as it effectively captures the dynamics of convergence and tracking performance within the ILC system. Hence, we refer to this model as the equivalent 2D model for the ILC system.

Now, consider a 2D state feedback controller:

$$r(t,k) = G \begin{bmatrix} d(x(t,k)) \\ e(t+1,k-1) \end{bmatrix} \tag{9}$$

where $G \in \mathbb{R}^{m\times(n+l)}$ is the feedback gain matrix to be determined. Then, combining the 2D model Σ in (7) and the feedback controller in (9) yields the closed-loop model:

$$\begin{aligned} \Sigma_c: & \begin{bmatrix} d(x(t+1,k)) \\ e(t+1,k) \end{bmatrix} \\ & = (\bar{A}+\bar{B}G+\delta\bar{A}+\delta\bar{B}G) \begin{bmatrix} d(x(t,k)) \\ e(t+1,k-1) \end{bmatrix} \\ & \quad + \bar{D}w(t,k). \end{aligned} \tag{10}$$

This 2D model encapsulates the system's convergence and tracking performance. It also considers the uncertainties within the system parameters and provides a framework for potential algorithm development.

2.3 Metric for Bounding Error

It is necessary to monitor the variation of tracking errors to ensure the system's desired behavior in the presence of iteration-varying disturbances. Our objective is to minimize its sensitivity to such disturbances. Hence, a specific metric is defined as:

$$\gamma = \max_{k\in\mathbb{Z}^+} \frac{\sum_{t=1}^{T+1} \|e(t,k)\|^2 - \sum_{t=1}^{T+1} \|e(t,k-1)\|^2}{\sum_{t=0}^{T} \|w(t,k)\|^2} \tag{11}$$

where γ represents the maximum ratio of the variation in tracking error energy to the disturbance energy across all iterations.

In essence, γ provides a measurable way to see how the tracking error's variation responds to disturbances in the system. Reducing γ helps lessen the impact of disturbances, improving the system's stability and strengthening the system's robustness. This is important in engineering applications that require precise control when faced with iteration-varying disturbances. Note that a negative γ would imply that the tracking error's norm decreases as the number of iterations increases regardless of the disturbance, which is a strongly desired property for uncertain systems: robust monotonic convergence of tracking error.

A natural goal is to find the feedback gain G that minimizes metric γ. In this work, we provide an LMI condition for simultaneously finding an upper bound of γ and the associated G. Consider the following inequality for any $k \in \mathbb{Z}^+$:

$$\frac{\sum_{t=1}^{T+1} \|e(t,k)\|^2 - \sum_{t=1}^{T+1} \|e(t,k-1)\|^2}{\sum_{t=0}^{T} \|w(t,k)\|^2} \le \gamma. \quad (12)$$

Theorem 1. For any $k \in \mathbb{Z}^+$, inequality (12) is guaranteed if there exists a scalar $\varepsilon > 0$ such that the following LMI holds:

$$\begin{bmatrix} -\mathrm{diag}(I, \gamma I) & * & * \\ \begin{bmatrix} \bar{A}+\bar{B}G & \bar{D} \end{bmatrix} & \varepsilon^{-1}(\bar{E}_1\bar{E}_1^\top + \bar{E}_2\bar{E}_2^\top) - I & * \\ \begin{bmatrix} \bar{F}_1 & \mathbf{0} \\ \bar{F}_2 G & \mathbf{0} \end{bmatrix} & \mathbf{0} & -\varepsilon^{-1} I \end{bmatrix}$$

$$\preceq 0 \quad (13)$$

The proof can be found in Appendix B.

3 RESULTS

To solve LMI (13), we use the optimization toolbox YALMIP (Lofberg, 2004) and solver MOSEK (ApS, 2022). The key is to determine the decision variables G and ε in LMI (13) that minimize γ, which is the upper bound of the measure of the resilience of the tracking error to disturbance. We conduct a numerical simulation to validate the proposed strategy's effectiveness.

Consider the system represented by the following matrices:

$$A = \begin{bmatrix} 0.4 & -0.3 \\ 0.1 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 1 \end{bmatrix},$$

$$E_1 = \begin{bmatrix} 0.01 & 0.01 \\ 0 & 0 \end{bmatrix}, \ F_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$E_2 = \begin{bmatrix} 0.01 \\ 0 \end{bmatrix}, \ F_2 = 1. \quad (14)$$

The reference trajectory is given by:

$$y_r(t) = \sin(\frac{2\pi}{25}t), \quad t \in [0, 100]. \quad (15)$$

Solving LMI (13) with YALMIP and MOSEK gives a minimum γ value of 3.0033 with a feedback gain G of $[-0.4999, 0.29999, 0.6663]$ and an ε value of 0.41.

Figure 1 shows the Root Mean Square Error (RMSE) of tracking as the iteration changes. RMSE value obviously decreases as the number of iterations increases. However, the tracking error cannot converge perfectly to zero due to the iteration-varying disturbances. The residual error indicates that despite mitigations, the system experiences fluctuations that hinder precise tracking, likely due to the influence of disturbances that vary with each iteration.

Figure 2 shows the γ value as the iteration changes. The energy of the error may increase at some stages due to the presence of iteration-varying disturbances, but this effect (measured by γ) is limited to 3.0033 or less. This bounded increase in energy indicates that the control system is able to mitigate the impact of disturbances up to a certain level, maintaining overall stability and preventing excessive error growth.

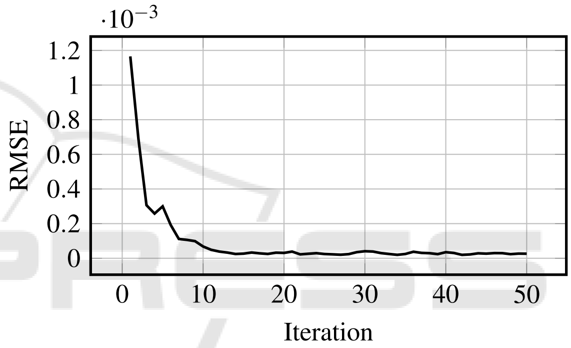

Figure 1: Tracking error convergence with iteration.

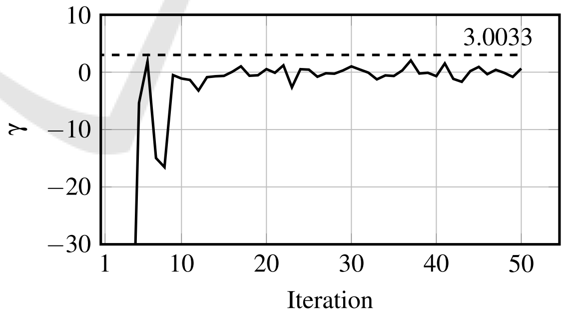

Figure 2: γ value variation with iterations.

4 CONCLUSIONS

In the field of ILC, addressing the challenges of time-varying modeling uncertainty and non-repeatable disturbances has been a persistent issue. This study introduces a unique metric and proposes an LMI condition to determine an upper bound for this metric based on the 2D Roesser modeling formulation. The Roesser model provides a robust framework for capturing the dynamics of processes that exhibit both temporal and iterative dependencies.

However, the proposed approach is limited to finding a positive upper bound of the metric, which does not guarantee robust monotonic convergence of the tracking error. Ensuring that the tracking error consistently decreases with each iteration is also important for robust ILC performance. Our future research direction will focus on addressing this limitation.

Additionally, while the theoretical analysis provides a solid foundation for our approach, further practical validation is necessary. Preliminary numerical simulation results are reasonable, demonstrating potential resistance to iteration-varying disturbances. This indicates that our approach can effectively handle variability and unpredictability, improving the robustness and reliability of the ILC system.

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APPENDIX A

Lemma 1. (Du and Xie, 1999) Assume A, E, F and $Q = Q^\top$ are given matrices with appropriate dimensions. For all matrix Δ, satisfying $\Delta\Delta^\top \leqslant I$, there exists a positive definite matrix $P \succ 0$ satisfying

$$(A + E\Delta F)^\top P(A + E\Delta F) - Q \prec 0 \tag{16}$$

if and only if there exist a scalar $\varepsilon > 0$ and a positive definite matrix $P \succ 0$ such that

$$\begin{bmatrix} -Q + \varepsilon F^\top F & A^\top P & \mathbf{0} \\ PA & -P & PE \\ \mathbf{0} & E^\top P & -\varepsilon I \end{bmatrix} \prec 0. \tag{17}$$

Lemma 2. (Boyd et al., 1994) Assume W, L and V are given matrices with appropriate dimensions, where W and V are positive definite symmetric matrices. Then

$$L^\top V L - W \prec 0 \tag{18}$$

if and only if

$$\begin{bmatrix} -W & L^\top \\ L & -V^{-1} \end{bmatrix} \prec 0 \tag{19}$$

or

$$\begin{bmatrix} -V^{-1} & L \\ L^\top & -W \end{bmatrix} \prec 0. \tag{20}$$

APPENDIX B

Assume the boundary condition is maintaining zero state transition between consecutive iterations at $t = 0$, which implies $d(x(0,k)) = x(0,k) - x(0,k-1) = 0$. This condition is trivial and can be ensured, for instance, by keeping the system state $x(0,k)$ unchanged across all iterations $k \in \mathbb{Z}^+$.

Recall that the upper bound inequality for γ is given by:

$$\frac{\sum_{t=1}^{T+1} \|e(t,k)\|^2 - \sum_{t=1}^{T+1} \|e(t,k-1)\|^2}{\sum_{t=0}^{T} \|w(t,k)\|^2} \leq \gamma. \tag{21}$$

Firstly, observe that inequality (21) is satisfied by:

$$\|d(x(T+1,k))\|^2 + \sum_{t=1}^{T+1} \|e(t,k)\|^2 \leq \|d(x(0,k))\|^2 + \sum_{t=1}^{T+1} \|e(t,k-1)\|^2 + \gamma \sum_{t=0}^{T} \|w(t,k)\|^2. \tag{22}$$

Given that $d(x(0,k)) = x(0,k) - x(0,k-1) = 0$ and $\|d(x(T+1,k))\|^2 \geq 0$, the above inequality holds.

Next, we show that inequality (22) can be guaranteed by the following system of inequalities:

$$\begin{aligned} \|d(x(1,k))\|^2 + \|e(1,k)\|^2 &\leq \|d(x(0,k))\|^2 + \|e(1,k-1)\|^2 + \gamma\|w(0,k)\|^2, \\ \|d(x(2,k))\|^2 + \|e(2,k)\|^2 &\leq \|d(x(1,k))\|^2 + \|e(2,k-1)\|^2 + \gamma\|w(1,k)\|^2, \\ &\vdots \\ \|d(x(T,k))\|^2 + \|e(T,k)\|^2 &\leq \|d(x(T-1,k))\|^2 + \|e(T,k-1)\|^2 + \gamma\|w(T-1,k)\|^2, \\ \|d(x(T+1,k))\|^2 + \|e(T+1,k)\|^2 &\leq \|d(x(T,k))\|^2 + \|e(T+1,k-1)\|^2 + \gamma\|w(T,k)\|^2. \end{aligned} \tag{23}$$

We obtain inequality (22) by summing the above system of inequalities (23). Notice that the system of inequalities (23) can be derived from:

$$\|d(x(t+1,k))\|^2 + \|e(t+1,k)\|^2 \leq \|d(x(t,k))\|^2 + \|e(t+1,k-1)\|^2 + \gamma\|w(t,k)\|^2, \quad t = 0,1,\ldots,T. \tag{24}$$

Construct the Lyapunov function as:

$$V\left(\begin{bmatrix} d(x(t+1,k)) \\ e(t+1,k) \end{bmatrix}\right) = \|d(x(t+1,k))\|^2 + \|e(t+1,k)\|^2 \tag{25}$$

and

$$V\left(\begin{bmatrix} d(x(t,k)) \\ e(t+1,k-1) \end{bmatrix}\right) = \|d(x(t,k))\|^2 + \|e(t+1,k-1)\|^2. \tag{26}$$

With the defined Lyapunov function, inequality (24) can be transformed into:

$$V\left(\begin{bmatrix} d(x(t+1,k)) \\ e(t+1,k) \end{bmatrix}\right) \leq V\left(\begin{bmatrix} d(x(t,k)) \\ e(t+1,k-1) \end{bmatrix}\right) + \gamma\|w(t,k)\|^2. \tag{27}$$

It follows from model (10) that inequality (27) can be transformed into:

$$\|(\bar{A} + \bar{B}G + \delta\bar{A} + \delta\bar{B}G)\begin{bmatrix} d(x(t,k)) \\ e(t+1,k-1) \end{bmatrix} + \bar{D}w(t,k))\|^2 \leq \|d(x(t,k))\|^2 + \|e(t+1,k-1)\|^2 + \gamma\|w(t,k)\|^2. \tag{28}$$

Rewrite inequality (28) in matrix form:

$$\begin{bmatrix} d(x(t,k)) \\ e(t+1,k-1) \\ w(t,k) \end{bmatrix}^\top \begin{bmatrix} (\bar{A} + \bar{B}G + \delta\bar{A} + \delta\bar{B}G)^\top(\bar{A} + \bar{B}G + \delta\bar{A} + \delta\bar{B}G) - I & * \\ \bar{D}^\top(\bar{A} + \bar{B}G + \delta\bar{A} + \delta\bar{B}G) & \bar{D}^\top\bar{D} - \gamma I \end{bmatrix} \begin{bmatrix} d(x(t,k)) \\ e(t+1,k-1) \\ w(t,k) \end{bmatrix} \preceq 0. \tag{29}$$

According to the definition of the negative semi-definite matrix, inequality (29) holds if and only if the matrix in the middle is negative semi-definite, that is,

$$\begin{bmatrix} (\bar{A} + \bar{B}G + \delta\bar{A} + \delta\bar{B}G)^\top(\bar{A} + \bar{B}G + \delta\bar{A} + \delta\bar{B}G) - I & * \\ \bar{D}^\top(\bar{A} + \bar{B}G + \delta\bar{A} + \delta\bar{B}G) & \bar{D}^\top\bar{D} - \gamma I \end{bmatrix} \preceq 0. \tag{30}$$

Substitute δA with $E_1\Delta F_1$ and δB with $E_2\Delta F_2$, where $\Delta^T\Delta \leq I$. After making these substitutions, inequality (30) transforms into:

$$\begin{bmatrix} (\bar{A} + \bar{B}G + \bar{E}_1\Delta\bar{F}_1 + \bar{E}_2\Delta\bar{F}_2G)^\top(\bar{A} + \bar{B}G + \bar{E}_1\Delta\bar{F}_1 + \bar{E}_2\Delta\bar{F}_2G) - I & * \\ \bar{D}^\top(\bar{A} + \bar{B}G + \bar{E}_1\Delta\bar{F}_1 + \bar{E}_2\Delta\bar{F}_2G) & \bar{D}^\top\bar{D} - \gamma I \end{bmatrix} \preceq 0. \tag{31}$$

Using Lemma 1, inequality (31) can be guaranteed if and only if there exists a scalar $\varepsilon > 0$ such that:

$$\begin{bmatrix} -\mathrm{diag}(I,\gamma I) + \varepsilon \begin{bmatrix} \bar{F}_1 & \mathbf{0} \\ \bar{F}_2G & \mathbf{0} \end{bmatrix}^\top \begin{bmatrix} \bar{F}_1 & \mathbf{0} \\ \bar{F}_2G & \mathbf{0} \end{bmatrix} & * & * \\ \begin{bmatrix} \bar{A} + \bar{B}G & \bar{D} \end{bmatrix} & -I & * \\ \mathbf{0} & \begin{bmatrix} \bar{E}_1 & \bar{E}_2 \end{bmatrix}^\top & -\varepsilon I \end{bmatrix} \preceq 0. \tag{32}$$

Then, by applying Lemma 2 (Schur complement), inequality (32) can be guaranteed if and only if:

$$\begin{bmatrix} -\mathrm{diag}(I,\gamma I) & * & * \\ \begin{bmatrix} \bar{A} + \bar{B}G & \bar{D} \end{bmatrix} & \varepsilon^{-1}(\bar{E}_1\bar{E}_1^\top + \bar{E}_2\bar{E}_2^\top) - I & * \\ \begin{bmatrix} \bar{F}_1 & \mathbf{0} \\ \bar{F}_2G & \mathbf{0} \end{bmatrix} & \mathbf{0} & -\varepsilon^{-1} I \end{bmatrix} \preceq 0. \tag{33}$$