An Approach for Fractional Commensurate Order Youla Parametrization Using $\overline{q}$-weighted Operator

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Abstract: The Youla-Kucera parametrization is a strategy widely used for robust control design and system identification of integer systems, but with the increasing interest in fractional order controllers, a new window for research and development is widely open. In this work this parametrization is extended to fractional commensurate order systems using the $\overline{q}$-weighted operator; originally developed for the field of theoretical physics, is proposed as a tool for developing robust fractional order controllers, the proposal is evaluated in two simulated processes and implemented in the TCLAB process.

1 INTRODUCTION

In industrial environments, controllers are essential for meeting performance criteria like stabilization, sensitivity, and robustness. Meeting all these requirements simultaneously is challenging, leading to the development of polynomial techniques for the parametrization of controllers that stabilize a given plant (Kučera, 2007). The parametrization proposed by (Youla et al., 1976) has been widely utilized across various applications, including closed-loop identification and adaptive control as found in(Anderson, 1998; Forssell and Ljung, 1999). Recently, Youla parametrization has been used in applications such as plug&play control and multi model adaptive control among others, (Mahtout et al., 2020).

In the other hand, with the increasing interest in industrial applications for fractional order controllers as shown in (Tepljakov et al., 2021) one main challenge is the implementation of such controllers, this is when the Youla Parametrization could serve as an useful answer to implement such controllers in the industrial field aided by the use of the $\overline{q}$-weighted operator (paper developed by the third author of the current article), originally developed to obtain a set of fractional Einstein field equations within 2+1 dimensional spacetime, in the area of control systems this operator aids in the design of stabilizing controllers in fractional order independently and no limited to first order plus time delay (FOPDT) processes as the methods shown in (Di Teodoro et al., 2022) and (Ranganayakulu et al., 2016); in other words, allows the design of different types of controllers including fractional PID controllers for systems with models different than the FOPDT.

In this first approach in the use of the $\overline{q}$-weighted operator and Youla Parametrization this paper is structured as follows: Section 2 presents a brief introduction to Fractional Calculus in which fundamentals of the $\overline{q}$-weighted operator are presented, its relation with the Youla Parameter and the basis for controller design are presented in Section 3, followed by a set of applications using the toolbox FOMCON (Tepljakov and Tepljakov, 2017) in Section 4 including a real time implementation in TCLAB from apmonitor

2 BRIEF INTRODUCTION TO FRACTIONAL CALCULUS

Definition 2.1. The Riemann–Liouville fractional integral of order $\eta > 0$ is given by (see (Kilbas et al., 2006; Podlubny, 1994; Kilbas et al., 1993))

$$\left(I^{\eta}_{a^+}h\right)(x) = \frac{1}{\Gamma(\eta)}\int_a^x \frac{h(t)}{(x-t)^{1-\eta}}\,dt, \quad x > a. \quad (2.1)$$

We denote by $I^{\eta}_{a^+}(L_1)$ the class of functions h, represented by the fractional integral (2.1) of a summable function, that is $h = I^{\eta}_{a^+}\varphi$, where $\varphi \in L_1(a,b)$. A description of this class of functions was provided in

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(Kilbas et al., 2006; Kilbas et al., 1993; Stein and Shakarchi, 2009).

Theorem 2.2. *A function $h \in I^{\eta}_{a^+}(L_1), \eta > 0$, if and only if its fractional integral $I^{s-\eta}_{a^+} h \in AC^s([a,b])$, where $s = [\eta]+1$ and $(I^{s-\eta}_{a^+} h)^{(k)}(a) = 0$, for $k = 0, \ldots, s-1$.*

In Theorem 2.2, $AC^s([a,b])$ denotes the class of functions h, which are continuously differentiable on the segment $[a,b]$, up to order $s-1$ and $h^{(s-1)}$ is absolutely continuous on $[a,b]$. By removing the last condition in Theorem 2.2, we obtain a class of functions that admit a summable fractional derivative. (See (Kilbas et al., 2006; Kilbas et al., 1993))

Definition 2.3 (see (Kilbas et al., 1993))**.** A function $h \in L_1(a,b)$ has a summable fractional derivative $\left(D^{\eta}_{a^+} h\right)(x)$ if $\left(I^{s-\eta}_{a^+} h\right)(x) \in AC^s([a,b])$, where $s = [\eta]+1$.

Definition 2.4. Let $\left(D^{\eta}_{a^+} h\right)(x)$ denote the **fractional Riemann–Liouville derivative** of order $\eta > 0$ (see (Kilbas et al., 2006; Podlubny, 1994; Kilbas et al., 1993))

$$\begin{aligned}\left(D^{\eta}_{a^+} h\right)(x) &= \left(\frac{d}{dx}\right)^s \frac{1}{\Gamma(s-\eta)} \int_a^x \frac{h(t)}{(x-t)^{\eta-s+1}}\, dt \\ &= \left(\frac{d}{dx}\right)^s \left(I^{s-\eta}_{a^+} h\right)(x), \end{aligned} \tag{2.2}$$

where $s = [\eta]+1, x > a$ $[\eta]$ denotes the integer part of η and Γ is the gamma function. When $0 < \eta < 1$, then (2.2) takes the form

$$\left(D^{\eta}_{a^+} h\right)(x) \;= \frac{d}{dx}\left(I^{1-\eta}_{a^+} h\right)(x). \tag{2.3}$$

Note that, when $\eta \to 1$, we recover the typical derivative operator (Kilbas et al., 2006; Podlubny, 1994; Kilbas et al., 1993; Ceballos et al., 2020).

The semigroup property for the composition of fractional derivatives does not hold generally (see (Podlubny, 1994, Sect. 2.3.6)). In fact, the property:

$$D^{\eta}_{a^+}\left(D^{\gamma}_{a^+} h\right) = D^{\eta+\gamma}_{a^+} h, \tag{2.4}$$

holds if

$$h^{(j)}(a^+) = 0, \qquad j = 0,1,\ldots,s-1, \tag{2.5}$$

and $h \in AC^{s-1}([a,b])$, $h^{(s)} \in L_1(a,b)$ and $s = [\gamma]+1$. Thus, we can write this result in the following:

Lemma 2.5. *Consider $h \in AC^{s-1}([a,b])$ and $h^{(s)} \in L_1(a,b)$ then,*

$$D^{\eta}_{a+}\left(D^{\gamma}_{a+}\right) h = D^{\gamma}_{a+}\left(D^{\eta}_{a+}\right) h, \tag{2.6}$$

holds whenever

$$h^{(j)}(a^+) = 0, \quad j = 0,1,\ldots,s-1, \tag{2.7}$$

where $s = [\gamma]+1$.

Proof. This proof can be found in (Podlubny, 1994, Secton 2.3.6). □

Remark 2.6. It is worth noticing that the **Riemann-Liouville derivative of a constant is not zero**. However, in the limit process, it behaves as expected.

$$\lim_{\eta\to 1}\left(D^{\eta}_{a^+} 1\right)(x) = \lim_{\eta\to 1} \frac{(x-a)^{-\eta}}{\Gamma(1-\eta)} = 0. \tag{2.8}$$

Example. Consider $\alpha \in (0,1)$, $a^+ > 0$ and for $m \in \mathbb{N}$ (See (Kilbas et al., 2006; Kilbas et al., 1993; Ceballos et al., 2022)).

$$\left[D^{\alpha}_{a^+}(t-a)^{(m+1)\alpha-1}\right] = (t-a)^{m\alpha-1},$$
$$\left[D^{\alpha}_{a^+}(t-a)^{\alpha-1}\right] = 0 \text{ if } \alpha < 1 \text{ and } x > 0.$$

There are other types of derivatives, such as the Caputo derivative (where the derivative of a constant is zero), the Caputo-Fabrizio derivative, the Hilfer derivative, among others. However, for this proposal, we will use a modification of the Riemann operator that recently has applications in physics, specifically in developing solutions for Fractional Einstein field equations. This is our primary motivation: starting from an operator that has applications in physics and exploring how it can contribute to control theory.

Now we will introduce the weighted fractional operator, the central piece of all our subsequent development.

Definition 2.7. Consider $q_1(x,\eta)$ a continuous function, $q_2(x,\eta)$ a continuously differentiable function on x and let $\left({}^{(q_1,q_2)}D^{\eta}_{a^+} h\right)(x) = \left({}^{\overline{q}}D^{\eta}_{a^+} h\right)(x)$ denote the **$\overline{q}$-weighted fractional Riemann-Liouville derivative** of order $\eta > 0$. For $q_1, q_2 \in AC^s(\mathbb{R})$

$$\left({}^{\overline{q}}D^{\eta}_{a^+} h\right)(x) = q_1(x,\eta)\left(\frac{d}{dx}\right)^s q_2(x,\eta)\left(I^{s-\eta}_{a^+} h\right)(x), \tag{2.9}$$

where $s = [\eta]+1, x > a$ and $[\eta]$ denotes the integer part of η.

Remark 2.8. To recover the Riemann-Liouville operator defined in 2.2, it is sufficient to take $\lim_{\eta\to 1} q_1(x,\eta) = \lim_{\eta\to 1} q_2(x,\eta) = 1$, and to obtain the classical derivative, it is enough to take $\alpha \to 1$.

Example. Consider $0 < \eta < 1$ and taking $q_2(x,\eta) = (x-a)^{\eta-1}$ so then (2.9) takes the form

$$\left({}^{\overline{q}}D^{\eta}_{a^+}h\right)(x) = q_1(x,\eta)\left(\frac{d}{dx}\right)(x-a)^{\eta-1}\left(I^{1-\eta}_{a^+}h\right)(x). \quad (2.10)$$

As a consequence of the form of q_2, we have that for any q_1, the derivative of a constant is zero

$$\left({}^{\overline{q}}D^{\eta}_{a^+}1\right)(x) = 0 \quad (2.11)$$

As in the case of the Riemann-Liouville operator, It is not difficult to see that the operator is linear and the semigroup property for the composition of fractional derivatives is generally not satisfied. Nonetheless, we can obtain similar relation
Based on the operator (2.9), and considering numerical implementations with much greater simplicity and computational ease, we slightly modify the structure of the operator, keeping the form, but now thinking of it in terms of convolutions.

$$\left({}^{\overline{q}}D^{\eta}_{a^+}h\right)(x) = q_1(t,\eta) * \frac{d}{dt}q_2(x,\eta) * \left(I^{1-\eta}_{a^+}h\right)(t) + q_1(t,\eta) * q_2(t,\eta) * \left(D^{\eta}_{a^+}h\right)(t) \quad (2.12)$$

Example. Let's consider in fig:1 the case where $\alpha = 0.5$ and the weights are defined as $q_1 = 1$ and $q_2 = (x-a)^{\alpha}$ with $a = 0$ in the Q operator as defined in 2.10. The weight q_2 is specifically chosen to cancel the Q operator when the function $f(t)$ is a constant.

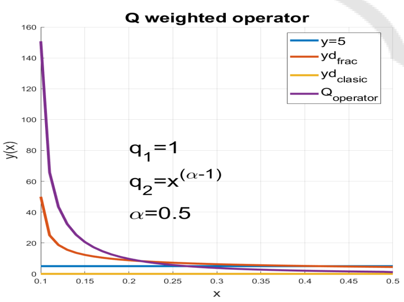

Figure 1: Example Q operator.

2.0.1 Laplace Transform of Fractional-Order Derivatives

Definition 2.9. Consider $F(s) := (\mathscr{L}f)(s)$ to represent the Laplace transform of certain function f, $\alpha \in (n-1,n]$, and $n \in \mathbb{N}$ (See (Kilbas et al., 1993)).

$$\left(\mathcal{L}\, D^{\alpha}_{0^+}f\right)(s) = s^{\alpha}F(s) - \sum_{k=1}^{n} s^{k-1}(D^{\alpha-k}_{0^+}f)(0)$$

It is clear that if $\alpha \in (0,1]$, then

$$\left(\mathcal{L} D^{\alpha}_{0^+}f\right)(s) = s^{\alpha}F(s) - (D^{\alpha-1}_{0^+}f)(0) \quad (2.13)$$

By applying the Laplace transform to the following modified $\overline{q}$-weighted operator with $0 < \eta < 1$.

$$\left({}^{\overline{q}}D^{\eta}_{a^+}h\right)(t) = q_1(t,\eta) * \frac{d}{dt}q_2(x,\eta) * \left(I^{1-\beta}_{a^+}h\right)(t) + q_1(t,\eta) * q_2(t,\eta) * \left(D^{\eta}_{a^+}h\right)(t) \quad (2.14)$$

we obtain:

$$\left(\mathcal{L}^{\overline{q}}D^{\eta}_{a^+}h\right)(s) = q_1(s,\eta)q_2(s,\eta)[s^{\beta} + s^{\alpha}]h(s) \quad (2.15)$$

2.1 The $\overline{q}$-weighted Grünwald-Letnikov Derivative Version

The Grünwald-Letnikov derivative is defined as (Palacios et al., 2023):

Definition 2.10. Let $\eta > 0$, $f \in C^k[a,b]$, and $a < x \leq b$. Then

$$\mathcal{G}^{\eta}_{a}f(x) = \lim_{N\to\infty}\frac{\Delta^{\eta}_{h_N}f(x)}{h^{\eta}_{N}} = \lim_{h\to 0}\frac{1}{h^{\beta}}\sum_{k=0}^{\infty}(-1)^k\binom{\beta}{k}f(x-kh), \quad (2.16)$$

with $h = \frac{(x-a)}{N}$, $N = 1,2,\ldots$.
This definition involves a sum of values of the function $f(x)$ at different points.

Remark 2.11. It is worth mentioning that the Grünwald-Letnikov derivative can be seen as a discrete approximation of the Riemann-Liouville derivative, and in the limit, as the step size goes to zero, the Grünwald-Letnikov derivative becomes a continuous fractional derivative.

Having the definition of the Grünwald-Letnikov, we can define our version of the weighted operator, which will be used for the numerical implementations in this work.

The Grünwald-Letnikov $\overline{q}$-weighted version

$$\left({}^{\overline{q}}\mathcal{G}^{\eta}_{a^+}h\right)(t) = q_1(t,\eta) * \frac{d}{dt}q_2(x,\eta) * \left(I^{1-\eta}_{a^+}h\right)(t) + q_1(t,\eta) * q_2(t,\eta) * \left(\mathcal{G}^{\eta}_{a^+}h\right)(t) \quad (2.17)$$

Example. Let's consider the case in fig:2 where $\alpha = 0.5$ and the weights are defined as $q_1 = (x-a)^{\alpha}$ and $q_2 = 1$, with $a = 0$. The function $f[t] = x^2$is chosen to illustrate how the Qweighted operator functions in a specific scenario. The analytical implementation defined in 2.10 is contrasted with the Grünwald–Letnikov approximation defined in 2.17, revealing minor differences between the two approaches.

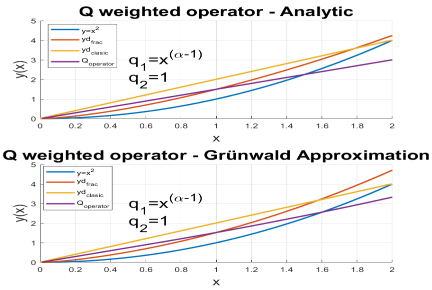

Figure 2: Example Q operator.

3 THE $\overline{q}$-weighted OPERATOR AND THE RELATION WITH YOULA PARAMETRIZATION

In the previous section the generalized $\overline{q}$-weighted operator is presented, in this section, the operator is used for fractional order control.

In order to further explain the proposed method, it is necessary to keep in mind some necessary basic concepts that are formalized in the following paragraphs.

Definition 3.1. A system $P(s^\alpha) = \frac{N(s^\alpha)}{D(s^\alpha)}$ is said to be *proper* if the degree of $D(s^\alpha)$ is larger or equal than the degree of $N(s^\alpha)$(Goodwin et al., 2001).

Definition 3.2. A system $P(s^\alpha)$ is said to be commensurate if the dynamics order of the fractional system are equal; otherwise, is incommensurate (Tavazoei and Asemani, 2020).

Definition 3.3. A system is said to be *BIBO stable* (bounded-input bounded-output) if every bounded input excites a bounded output. This stability is defined for the zero-state response and is applicable only if the system is initially relaxed (Chen, 1999).

Theorem 3.4. *(Petráš and Petráš, 2011) A fractional commensurate system* $P(s^\alpha) = \frac{N(s^\alpha)}{D(s^\alpha)}$ *is said to be stable if and only if considering a fractional operator* $\alpha \in (0,2)$ *then*

$$|\, arg[eig(P(s^\alpha))]\,| > \alpha\frac{\pi}{2} \tag{3.1}$$

Definition 3.5. A fractional incommensurate system $G(s^\beta)$ is said to be *BIBO stable* if and only if $\alpha \in (0,2)$ satisfy the following inequality (Petráš and Petráš, 2011):

$$|\, arg[eig(G(s^\beta))]\,| < \pi\left(1-\frac{\beta}{2}\right) \tag{3.2}$$

Two definitions for stability are presented in definitions 3.3,and 3.5 and one Theorem in 3.4 for control system purposes;with all this in mind, consider for instance the closed loop system shown in figure 3,where $r(t)$,$y(t)$, $u(t)$ and $n(t)$ are the reference signal, the output signal, control signal and the noise signal (white noise) respectively. P_R is the real process which for the moment is assumed stable in open loop and $C(s^\alpha)$ is the fractional controller to be estimated.

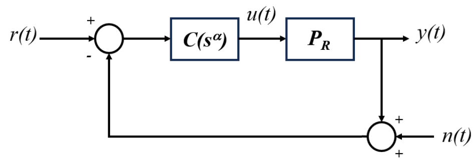

Figure 3: Closed loop linear fractional system with noise.

Following (Mahtout et al., 2020) The set of transfer functions from $n(s)$ and $r(s)$ to $y(s)$ are shown as follows:

$$y(s) = \frac{C(s^\alpha)P_R}{1+C(s^\alpha)P_R}r(s) - \frac{C(s^\alpha)P_R}{1+C(s^\alpha)P_R}n(s), \tag{3.3}$$

Now, the set of transfer functions from $r(s)$,$n(s)$ to $u(s)$ are estimated as follows:

$$u(s) = \frac{C(s^\alpha)}{1+C(s^\alpha)P_R}r(s) - \frac{C(s^\alpha)}{1+C(s^\alpha)P_R}n(s). \tag{3.4}$$

Following (Keviczky and Bányász, 2007; Mahtout et al., 2020) and adapting to this work it is clear that:

$$Q(s^\alpha) = \frac{C(s^\alpha)}{1+C(s^\alpha)P_R} \tag{3.5}$$

$Q(s^\alpha)$ in (3.5) is the Fractional Youla Parameter (Aboukheir et al., 2006; Aboukheir, 2010; Keviczky and Bányász, 2007; Mahtout et al., 2020).
Clearing for $C(s^\alpha)$ in (3.5) it is possible to obtain

$$C(s^\alpha) = \frac{Q(s^\alpha)}{1-Q(s^\alpha)P_R} \tag{3.6}$$

Replacing (3.6) in (3.3) following (Bars and Keviczky, 2015) the closed loop is transformed into the Internal Model Structure shown as follows

$$y(s) = P_R Q(s^\alpha)[r(s)-n(s)] \tag{3.7}$$

The Youla Parameter $Q(s^\alpha)$ must be proper and stable (Valderrama et al., 2020), and selected with commensurate order according to (Petráš and Petráš, 2011) by the designer in order to guarantee the closed loop stability.
The control system presented in 3.7, is not the ideal approach when P_R is open loop unstable, or has a time delay; which is the case to be analyzed as follows, first consider again the system presented in figure 3 with

$P_R = P_0 e^{-tds}$ with P_0 a proper and stable open loop model coupled with a time delay td, the closed loop set of transfer functions from setpoint to output and setpoint to control signal are shown as follows:

$$y(s) = \frac{C(s^\alpha)P_0 e^{-tds}}{1 + C(s^\alpha)P_0 e^{-tds}}[r(s) - n(s)] \quad (3.8)$$

$$u(s) = \frac{C(s^\alpha)}{1 + C(s^\alpha)P_0 e^{-tds}}[r(s) - n(s)] \quad (3.9)$$

It is clear from (3.9) that $Q(s^\alpha)$ is:

$$Q(s^\alpha) = \frac{C(s^\alpha)}{1 + C(s^\alpha)P_0 e^{-tds}} \quad (3.10)$$

Rearranging for $C(s^\alpha)$ in (3.10) it is possible to obtain:

$$C(s^\alpha) = \frac{Q(s^\alpha)}{1 - Q(s^\alpha)P_0 e^{-tds}} \quad (3.11)$$

It is possible to replace $P_0 e^{-tds}$ in (3.11) with:

$$C(s^\alpha) = \frac{Q(s^\alpha)}{1 - Q(s^\alpha)[P_0 e^{-tds} - P_0 e^{-tds} + P_0]} \quad (3.12)$$

This modification is the well known Smith Predictor (Smith and Corripio, 2005) shown in Figure 4, which leads to the following parametrization of controller $C(s^\alpha)$:

$$C(s^\alpha) = \frac{Q(s^\alpha)}{1 - Q(s^\alpha)P_0} \quad (3.13)$$

Which is similar to (3.6) with the difference that the open loop model in (3.13) does not consider the full dynamic of the plant as P_R but only the stable open loop part without delay P_0.

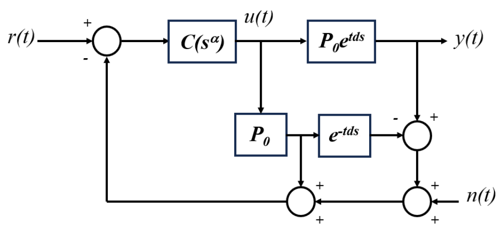

Figure 4: Closed loop system with Smith Predictor.

In this paper it is proposed to use the $\overline{q}$-weighted operator presented in (Contreras,) as the Youla Parameter. Using the laplace transform with zero initial conditions presented in (2.15) and selecting $f(s) = \lambda e(s)$ with $e(s)$ the error signal between the output and the reference and λ a parameter selected by the designer it is possible to obtain:

$$Q(s^\alpha) = q_1(s)q_2(s)[s^\alpha + s^\beta]\lambda e(s) \quad (3.14)$$

This is formalized in definition 3.6.

Definition 3.6. The group of controllers $C(s^\alpha)$ that stabilize $P_R = P_0$ and/or $P_0 e^{-tds}$ are parameterized by

$$C(s^\alpha) = \frac{Q(s^{\alpha)}}{1 - Q(s^{\alpha)}P_0} \quad (3.15)$$

with $Q(s^{\alpha)}$ as presented in (3.14) must be proper, commensurate order and stable.

Theorem 3.7. *An open loop system represented with the model $P_R = P_0$ or $P_0 e^{-tds}$ is stabilized in closed loop by $C(s^\alpha)$ if it exist an operator with commensurate order $Q(s^\alpha)$ proper and stable that parameterize $C(s^\alpha)$ according to (3.6) and relocates the closed loop poles of the system in such a way that:*

$$|\, arg[eig(Q(s^\alpha))]\,| \equiv |\, arg\left[eig\left(\frac{C(s^\alpha)P_R}{1 + C(s^\alpha)P_R}\right)\right] | > \alpha\frac{\pi}{2} \quad (3.16)$$

with $\alpha \in (0, 2)$

Proof. Consider an open loop system P_R stable and invertible and $C(s^\alpha)$ the unknown controller to be estimated,the closed loop transfer function is:

$$y(s) = \frac{P_R C(s^\alpha)}{1 + P_R C(s^\alpha)} r(s) \quad (3.17)$$

The parameter $Q(s^\alpha)$ is selected with commensurate order, proper and stable according Theorem 3.4 and Definition 3.1 as follows:

$$Q(s^\alpha) = q_1(s)q_2(s)[s^\alpha + s^\beta]\lambda \quad (3.18)$$

with $q_1(s)$ selected as (Bars and Keviczky, 2015):

$$\lambda q_1(s)[s^\alpha + s^\beta] = [P_R]^{-1} \quad (3.19)$$

and $q_2(s)$ fulfilling Theorem 3.4 as:

$$q_2(s) = \frac{1}{s^{n\alpha} + a_1 s^{(n-1)\alpha} + \ldots + a_n s^\alpha + 1} \quad (3.20)$$

The proper and stable $Q(s^\alpha)$ is represented as:

$$Q(s^\alpha) = \frac{[P_R]^{-1}}{s^{n\alpha} + a_1 s^{(n-1)\alpha} + \ldots + a_n s^\alpha + 1} \quad (3.21)$$

using (3.6) or (3.13) it is possible to obtain the controller $C(s^\alpha)$:

$$C(s^\alpha) = \frac{[P_R]^{-1}}{s^{n\alpha} + a_1 s^{(n-1)\alpha} + \ldots + a_n s^\alpha} \quad (3.22)$$

introducing the controller $C(s^\alpha)$ in 3.17 it is possible to obtain the controlled closed loop transfer function:

$$y(s) = \frac{1}{s^{n\alpha} + a_1 s^{(n-1)\alpha} + \ldots + a_n s^\alpha + 1} r(s) \quad (3.23)$$

It is clear that the stabilized poles of the closed systems are the same as the poles of $Q(s^\alpha)$ □

Theorem 3.7 it is possible to extract the following observation formalized in Lemma 3.8

Lemma 3.8. *The value of* α *and* β *in (3.14) are related with the zeros of* $Q(s^\alpha)$ *in order to fulfill theorem* 3.7 *it is necessary that* $\alpha \equiv \beta$.

4 APPLICATIONS

4.1 First Experiment

For the first experiment is the mixing process, a hot water stream $F_1(t)$ is manipulated to mix with a cold water stream $F_2(t)$ to obtain an output flow $F_0(t)$ at the desired temperature $T_0(t)$. The temperature transmitter is located at a distance L from the mixing tank bottom. This highly nonlinear system has variable dynamic see (Aboukheir et al., 2021) for details.

For a given operating point, it is estimated the following first order plus dead time model (Aboukheir et al., 2021):

$$P_R(s) = \frac{0.38}{6.41s+1}e^{-25s} \tag{4.1}$$

It is clear from (4.1) that a Smith Predictor is required for the control calculations, using the scheme presented in figure 4, and taking into the account that the system is filled up of uncertainties in gain, time constant and time delay, it is necessary to build a controllers that rejects disturbances while guaranteeing setpoint tracking, using (3.13) the value of $\alpha = \beta = 0.5$ and $\lambda = 0.5$ is selected with $q_1(s)$ and $q_2(s)$:

$$q_1(s) = 0.5\left[6.41^{0.5} + \frac{1}{s^{0.5}}\right]2s^{0.5} = 6.41s+1 \tag{4.2}$$

$$q_2(s) = \frac{1}{142.85s^{1.1}+0.38} \tag{4.3}$$

The proper and stable $Q(s^{\alpha})$ is presented for this system according Definitions 3.1 and Theorem 3.4 as follows:

$$Q(s^{\alpha}) = \frac{6.41s+1}{142.85s^{1.1}+0.38} \tag{4.4}$$

With this in mind, the controller $C(s^{\alpha})$ is:

$$C(s^{\alpha}) = \frac{0.007(6.41s+1)}{s^{1.1}} \tag{4.5}$$

The fractional controller obtained in (4.5) is compared with a Fractional PI, tuned using Chen tuning rules found in (Ranganayakulu et al., 2016), obtaining the following controller:

$$FPID(s) = 0.9916 + \frac{0.0615}{s^{1.1}} \tag{4.6}$$

The output of the closed loop system with the FPID is shown in figure 5, where it is possible to see that the Fractional PID cannot handle the uncertainties of the system, this result is similar than the Integer PID found in (Aboukheir, 2023)

In figure 6 the controller $C(s^{\alpha})$ using the proposed method is tested against the inherent uncertainties of the system adding some large measurables disturbances and unmeasurable disturbances in the form of white noise. The output of the controller is shown in figure 7

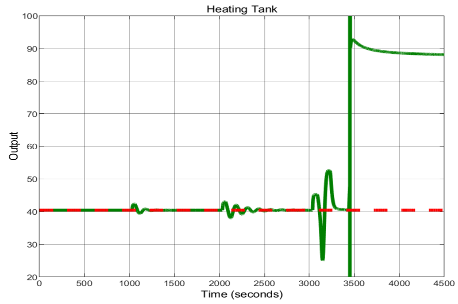

Figure 5: Closed loop response of the system with FPID(s).

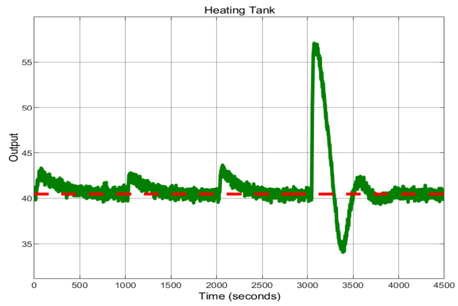

Figure 6: Closed loop response of the system with $C(s^{\alpha})$.

From figure 6 it is clear that the controller can handle the large uncertainties and the variability in the time delay, rejecting the measurables and unmeasurables disturbances while keep the system following the setpoints, but is in 7 when it is possible to observe that the controller filters the noise, providing a filtered control signal, in the following experiment, the proposed fractional controller is implemented in a real system.

4.2 Second Experiment

The second experiment,is the temperature control lab (TCLAB) from APMONITOR, this process is connected with Matlab/Simulink where the proposed controller is installed, figure 8 shows the connection used for this test.

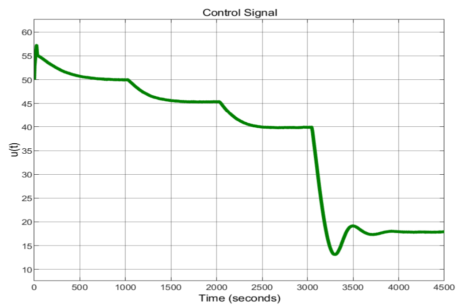

Figure 7: Output of the Fractional Controller $C(s^{\alpha})$.

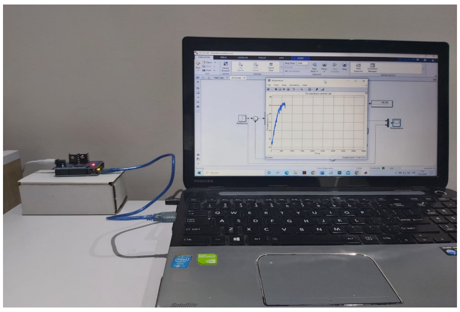

Figure 8: Temperature control lab connected with simulink.

An estimated linear model is obtained for this nonlinear system, which is presented as follows:

$$y(s) = \frac{0.8265}{165s+1} e^{-25s} u(s) \tag{4.7}$$

This system has a delay so, a controller with smith predictor is selected as the one presented in figure 4, following the procedure shown in the previous examples, fulfilling definitions 3.1 and Theorem 3.4 $Q(s^{\alpha})$ and $C(s^{\alpha})$ are respectively:

$$Q(s^{\alpha}) = \frac{(165s+1)}{285.71s^{1.1}+0.8265} \tag{4.8}$$

$$C(s^{\alpha}) = 0.0035\frac{(165s+1)}{s^{1.1}} \tag{4.9}$$

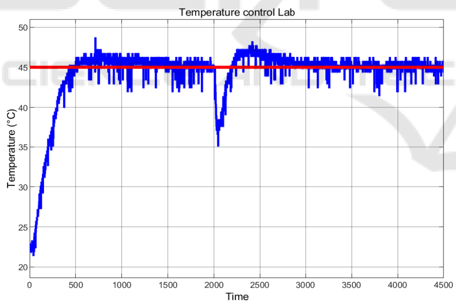

Figure 9: Closed loop system with $C(s^{\alpha})$ (red) Setpoint (blue) Implemented closed loop system.

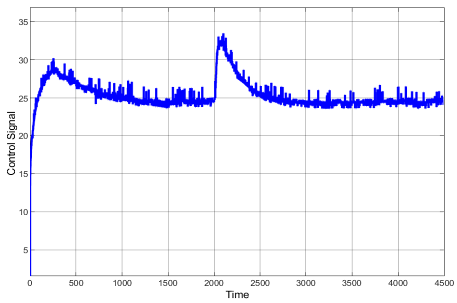

Figure 10: Output of the fractional controller $C(s^{\alpha})$.

From figures 9 and 10 it is clear that the fractional controller $C(s^{\alpha})$ must deal with uncertainties, measurable and unmeasurables disturbances, the robustness of the proposed controller minimize the effects of these elements while provides setpoint tracking throughout the whole operating region. From these experimnents it is clear that it is possible to parametrize controllers using Youla Parametrization through the use of the $\overline{q}$-weighted operator.

5 CONCLUSIONS

In this paper a methodology for Youla parametrization of fractional commensurate order controllers is presented; first a model of the process is required, with this model and the specified performance criteria the Youla parameter is built $Q(s^{\alpha})$ using the $\overline{q}$-weighted operator, this parametrization allows to design a robust loop controller that deals with noise, measurable disturbances and uncertainties while provides setpoint tracking as shown in the previously presented experiments, in future works the proposal is going to be extended to incommensurate systems.

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