Computing Bounds for the Synchronization Errors of Nonidentical Nonlinear Oscillators with Time-Varying Diffusive Coupling

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Abstract: The paper studies bounds of the synchronization error for networks of diffusively coupled and nonidentical nonlinear oscillators. In contrast to preceding work, which only analyzes the synchronization error in the limit and for constant coupling, a method based on over-approximating reachable sets of the synchronization error is proposed. The method allows the evaluation of different and time-varying gains along the limit cycles. Instead of using strong coupling gains to preserve synchronization, it is proposed to vary the coupling gains over time, while the synchronization error determined via reachable set does not exceed a specified bound. Evaluation results for the proposed method are provided for different types of coupled oscillators.

1 INTRODUCTION

The synchronization of sets of oscillators has been investigated in context of several applications, such as high-precision motion control (Xiao and Zhu, 2006), power grids (Wang et al., 2016), or in biological systems (Kim et al., 2019).

An essential point is to investigate in how far the coupling strengths among the oscillators as well as the interaction topology affects the synchronization error over time. This error encodes (temporary or permanent) deviation of the amplitudes (or phases) of the oscillators. While for coupled identical linear oscillators, the dynamics of the synchronization error can be expressed explicitly (Scardovi and Sepulchre, 2008), the situation for sets of nonlinear oscillators is more involved, since the dynamics of the synchronization error cannot be computed in closed form. Existing work can roughly be categorized in approaches that aim to seek sufficiently large static gains of diffusive coupling to obtain convergence of the errors to zero in the limit, and to those where the coupling strength is varied over time – this paper is focused on the latter case.

Among the existing approaches for this case, (Zhao et al., 2012) determine time-varying coupling gains by treating the error of nonidentical nonlinear systems with Lyapunov-like functions. Although bounded synchronization is guaranteed by the coupling gains, the network structure is restricted to an undirected form and the error is only considered in the limit, potentially leading to conservatively large gains. In the work by (Wang et al., 2011), the procedure is similar but only identical nonlinear systems in discrete time are considered. Also, (Zhu et al., 2020) and (Panteley and Loría, 2017) use the construction of Lyapunov-like functions to synthesize coupling gains bounding the error. The disadvantage of these approaches is, apart from using possibly larger gains than necessary for synchronization, the lack of the possibility to compute the evolution of the synchronization error over time.

A different approach for bounded synchronization uses funnel control (Lee et al., 2022), where the synchronization error of nonidentical nonlinear systems is forced to a specific bound by different time-varying coupling gains. However, the evolution of the error over time is restricted by the funnel shape, i.e., it is not possible to allow a larger synchronization error for certain times if an application requires this. It is not possible to check which error bounds arise from given coupling gains.

In contrast to the aforementioned approaches, the method proposed in this paper offers the possibility to (i) analyze if the synchronization error stays beyond a certain threshold for given coupling gains, and (ii) to synthesize time-varying coupling gains (possibly different for any coupling of the network topology) while maintaining a chosen error bound. The syn-

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thesis of gains is particularly useful if a compromise between the synchronization error and the synchronization effort (through large coupling gains) has to be determined. A recent paper by (Trummel et al., 2023) has suggested to solve this task by numeric optimization, however, the computation of error bounds was not included there.

In order to establish the schemes for analysis and synthesis, the paper on hand uses the technique of computing over-approximations of reachable sets for nonlinear systems, see e.g. (Stursberg and Krogh, 2003), (Girard, 2005), (Rungger and Zamani, 2018) and adopts it to analyze the evolution of the synchronization error. According to the knowledge of the authors, this is the first exposition on using reachable sets to explicitly compute upper bounds of synchronization errors for coupled nonlinear oscillators, while the only similar work by (Sang and Zhao, 2020) limits the focus to linear time-delayed local dynamics.

After introducing into the problem setting in Sec. 2, the main part (Sec. 3) introduces the procedure for computing over-approximations of reachable sets for synchronization errors for given coupling gains, including an analysis of the complexity of the computation. The section includes also a modification of the procedure, in which the coupling gains are iteratively changed on certain time intervals in order to limit the upper bounds of the synchronization errors on the respective intervals. The effectiveness of the techniques is illustrated by numerical examples in Sec. 4, before a conclusion and an outlook on future work is provided in Sec. 5.

2 PROBLEM DESCRIPTION

With an index set $\mathcal{N} = \{1, \ldots, n\}$, consider a set of nonlinear dynamics:

$$\dot{x}_i(t) = f_i(x_i(t)), i \in \mathcal{N} \tag{1}$$

with state vector $x_i \in \mathbb{R}^{n_x}$, time variable $t \in \mathbb{R}$, and flow function $f_i : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$.

Assumption 1. *Let $f_i(x_i(t))$ be continuous in time, continuously differentiable with respect to $x_i(t)$, as well as bounded on each compact subset of $\mathbb{R}^{n_x}$. For the solution of* (1)*, let furthermore a unique limit cycle* [1] *exist, and let $x_i(t)$ converge to the limit cycle for any initialization.*

With this assumption, we refer to the set of dynamics according to (1) as *nonidentical nonlinear oscillators*. Obviously, one can expect different limit cycles for any i in the general case. Now consider an extension of (1) by a diffusive coupling law to:

$$\dot{x}_i(t) = f_i(x_i(t)) + \sum_{j \in \mathcal{N}_i} k_{ij}(t)(x_j(t) - x_i(t)) \in \mathbb{R}^{n_x} \tag{2}$$

with the subset of $\mathcal{N}_i \subseteq \mathcal{N}$ of indices referring to the oscillators coupled directly to oscillator i, and $k_{ij}(t) \in \mathbb{R}^{\geq 0}$ denotes different time-varying coupling gains. Any $k_{ij}(t)$ models the individual gain by which the state difference of j and i affects the oscillator i. Assume that $k_{ij}(t) \neq k_{ji}(t)$ is possible and that the coupling network is directed and strongly connected.

From (Lee and Shim, 2022) it is known that the solutions of diffusively coupled nonidentical nonlinear systems[2] converge for a sufficiently large $k_{ij}(t) = k = \text{const.} \forall i, j \in \mathcal{N}$ to the solution $s(t) \in \mathbb{R}^{n_x}$ of the averaged dynamics:

$$\dot{s}(t) = \frac{1}{n} \sum_{i \in \mathcal{N}} f_i(s(t)),\ s(t_0) = \frac{1}{n} \sum_{i \in \mathcal{N}} x_i(t_0). \tag{3}$$

It is shown by (Lee and Shim, 2022) that for any synchronization accuracy $\varepsilon > 0$, a threshold κ of the coupling gain k exists for which:

$$\limsup_{t \to \infty} \| x_i(t) - s(t) \| \leq \varepsilon,\ \forall i \in \mathcal{N} \tag{4}$$

for any $k \geq \kappa \in \mathbb{R}^{\geq 0}$. Since Eq. (3) represents the averaged dynamics of all oscillators and encodes itself a nonlinear oscillator, the averaged trajectory $s(t)$ also converges to a limit cycle for $t \to \infty$.

In the following, the existence of the averaged dynamics is exploited to determine the error dynamics, whereas the requirement of a large gain is omitted, (e.g., in order to establish tradeoffs between synchronization errors and efforts). In particular, the quantitative determination of gains, or upper bounds of the synchronization errors are addressed along the following lines:

Problem 1. *For a given κ or given time-dependent profiles of the coupling gains $k_{ij}(t)$, determine the numeric value of the upper bound $\sigma > 0$ of the synchronization error.*

Problem 2. *For specified time-varying upper bounds of the synchronization error, determine values $k_{ij}(t)$ which ensure that error bounds are never exceeded.*

Both problems will be addressed based on the computation of over-approximations of the sets of reachable synchronization errors. When calculating the reachable sets, both the local dynamics and the

[1] For a definition of limit cycles, see e.g. the book by (Ye and Cai, 1986).

[2] (Lee and Shim, 2022) refer to the case that the network is undirected and connected, which corresponds to the case of a directed and strongly connected network.

network structure are taken into account in order to reduce the conservatism of the over-approximation. For the case of Problem 1, the question may arise where from the course of the $k_{ij}(t)$ would be known. A possible option is the use of the method by (Trummel et al., 2023), which uses a scheme of numeric optimization to compute the time-varying gains, however, without guaranteeing synchronization for these gains. The procedure proposed below to solve Problem 1 aims at providing this guarantee. The scheme to be developed as solution to Problem 2 iterates over candidates for the gain profiles in order to eventually satisfy the specified bounds on the synchronization error.

3 ENCLOSURE OF SYNCHRONIZATION ERRORS

3.1 Conservative Linearization of the Coupled Dynamics

Since it is, in the general case, not possible to compute the dynamics of the synchronization error in closed form, the concept followed here is to (1.) conservatively linearize the nonlinear oscillator dynamics, i.e., to resort to a local differential inclusions of affine structure (which is guaranteed to include the nonlinear dynamics), and (2.) to use the differential inclusions to determine upper bounds for the synchronization error.

Assume for the moment that all coupling gains $k_{ij}(t)$ would be known and constant at any time, such that a limit cycle of the averaged dynamics $s(t)$ will be obtained. It is then reasonable to select a suitable number of supporting points along this limit cycle in order to obtain the local conservative linearizations. Assume further that the oscillators have initial states $x_i(t_0) = x_{i,0}$ close to a point $s(t_0)$ that belongs to the solution of the averaged dynamics (i.e., an initial synchronization error is possible).

Choosing a first linearization point $x_{i,0}$ and an element of the set:

$$\mathcal{X}_{i,0} := \left\{ x \in \mathbb{R}^{n_x} \;\middle|\; |x - x_{i,0}| \leq \frac{\varepsilon}{\sqrt{n_x}} \mathbf{1}_{n_x} \right\} \quad (5)$$

for all $i \in \mathcal{N}$, where $\mathbf{1}_{n_x} \in \mathbb{R}^{n_x}$ denotes a vector of ones and ε is a given desired accuracy according to (4). Additional linearization points $x_{i,1}$, $x_{i,2}$, $\dots$, $x_{i,p}$ are distributed along the limit cycle $s(t)$ for sample times t_q, thus leading to partitioning of the period of the limit cycle into time intervals $[t_q, t_{q+1}]$ with $t_{q+1} = t_q + \delta t_q$ and $q \in \{0, 1, \dots, p-1\}$. $\delta t_q > 0$ can change from time step to time step, and is suitably chosen based on the curvature of $s(t)$, i.e., based on the value of the second derivative of (3).

Assumption 2. *Assume from now on that the coupling gains $k_{ij}(t)$ are piecewise constant on $[t_q, t_{q+1}]$ and thus denoted by $k_{ij,q}$.*

The linearization points result from the step-by-step computation of the solutions $x_i(t)\, \forall t \in [t_q, t_{q+1}]$, where the set $\mathcal{X}_{i,q}$ of the current linearization point is the initial set for the next computation. The set-based computation of the solutions is carried out using the procedure previously proposed in (Althoff et al., 2008), which uses Lagrangian remainders to conservatively encode the right-hand sides of (2). When denoting the Lagrangian remainders by $\mathcal{L}_{i,q,r}$ the conservative linearization of (2) around $x_{i,q}$ for all $i \in \mathcal{N}$ and $q \in \{0, 1, \dots, p-1\}$ is:

$$\begin{aligned} \dot{x}_{i,r}(t) = & f_{i,r}(s_0) + \frac{\partial f_{i,r}(x_i(t))}{\partial x_i(t)}\Big|_{x_i(t)=x_{i,q}} (x_i(t) - x_{i,q}) \\ & + \sum_{j \in \mathcal{N}_i} k_{ij,q}(x_{j,r}(t) - x_{i,r}(t)) + \mathcal{L}_{i,q,r} \end{aligned} \quad (6)$$

for each dimension $r \in \{1, \dots, n_x\}$ of $x_i(t)$ with $t \in [t_q, t_{q+1}]$. $\mathcal{L}_{i,q,r}$ is evaluated on a set $\mathcal{X}_{i,q}$ which is parameterized according to (5) using $x_{i,q}$, leading to the bound:

$$|\mathcal{L}_{i,q,r}| \leq w_{\max,i,q,r} \quad (7)$$

with:

$$w_{\max,i,q,r} := \max_{0 \leq \alpha \leq 1, x_i \in \mathcal{X}_{i,q}} \Big| \frac{1}{2}(x_i - x_{i,q})^T \frac{\partial^2 f_{i,r}(x_i)}{\partial x_i^2}\Big|_{x_i = \xi} (x_i - x_{i,q}) \Big|$$

and $\xi := x_{i,q} + \alpha(x_i - x_{i,q})$. The values $w_{\max,i,r}$ form a vector $w_{\max,i} \in \mathbb{R}^{n_x}$ and are used to establish:

$$\mathcal{W}_{\max,i,q} := \{ w_i \in \mathbb{R}^{n_x} \mid |w_i| \leq w_{\max,i,q} \}.$$

Now, the nonlinear dynamics (2) is over-approximated by the linearized inclusions:

$$\begin{aligned} \dot{x}_i(t) \in & (A_{i,q} x_i(t) + b_{i,q} \\ & + \sum_{j \in \mathcal{N}_i} k_{ij,q}(x_j(t) - x_i(t))) \oplus \mathcal{W}_{\max,i} \end{aligned} \quad (8)$$

for $t \in [t_q, t_{q+1}]$, where:

$$A_{i,q} := \left[\frac{\partial f_i(x_i)}{\partial x_i} \right] \Big|_{x_i = x_{i,q}} \text{ and } b_{i,q} := f_i(x_{i,q}) - A_{i,q} x_{i,q}$$

holds, and the symbol $\oplus$ denotes Minkowski addition. In order to ensure the tightness of the over-approximations, it is important to determine or select appropriate δt_q, as already mentioned.

Furthermore, the quantities of the local linearizations are collected in $A_q := \text{diag}(A_{1,q}, \dots, A_{n,q})$, $b_q := [b_{1,q}^T \dots b_{n,q}^T]^T$, $w_{m,q} := [w_{\max,1,q}^T \dots w_{\max,n,q}^T]^T$, and in the weighted Laplacian matrix:

$$K_q = \begin{bmatrix} m_{11} & m_{12} & \dots & m_{1n} \\ m_{21} & m_{22} & \dots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \dots & m_{nn} \end{bmatrix} \in \mathbb{R}^{n \times n}. \quad (9)$$

In here, $m_{ii} = \sum_{j \in \mathcal{N}_i} k_{ij,q}$ and $m_{ij} = -k_{ij,q}$ applies if a coupling from oscillator j to i exists, while $m_{ij} = 0$ otherwise. By defining $\mathcal{W}_{m,q} := \{w \in \mathbb{R}^{n_x n} \big| |w| \leq w_{m,q}\}$, the dynamics of the global vector $x(t) = [x_1^T(t) \dots x_n^T(t)]^T$ satisfies:

$$\dot{x}(t) \in ((A_q - (K_q \otimes I_{n_x}))x(t) + b_q) \oplus \mathcal{W}_{m,q} \quad (10)$$

for $t \in [t_q, t_{q+1}]$, where $\otimes$ denotes the Kronecker product and I_{n_x} is the $n_x \times n_x$ identity matrix.

3.2 Dynamics of the Synchronization Error

The synchronization error $e(t) \in \mathbb{R}^{n_x(n-1)}$ for $t \geq t_0$ is defined as the difference between $x_1(t)$ and any other $x_j(t)$, $j \in \{2, 3, \dots, n\}$, i.e.:

$$e(t) = \begin{bmatrix} x_2(t) - x_1(t) \\ x_3(t) - x_1(t) \\ \vdots \\ x_n(t) - x_1(t) \end{bmatrix} = (\underbrace{\begin{bmatrix} -1 & 1 & 0 & 0 & \dots & 0 \\ -1 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ -1 & 0 & \dots & 0 & 1 & 0 \\ -1 & 0 & \dots & 0 & 0 & 1 \end{bmatrix}}_{:=L_e} \otimes I_{n_x}) x(t).$$

By use of (10), the over-approximation of the reachable state set can be computed by propagating the set at time t_q forward over the time interval $[t_q, t_{q+1}]$. An over-approximation of reachable synchronization errors $e(t)$ can be determined by using the extreme values of each local state $x_i(t)$, $i \in \mathcal{N}$ in each dimension $r \in \{1, \dots, n_x\}$. The result may, however, be conservative, i.e., the set of feasible $e(t)$ may be considerably over-approximated. To address this problem, the conservatively linearized dynamics of $e(t)$ is determined according to (10): First, a new matrix:

$$L_d := \begin{bmatrix} [I_{n_x} \ \mathbf{0}_{n_x \times n_x(n-1)}] \\ L_e \end{bmatrix} \in \mathbb{R}^{n_x n \times n_x n} \quad (11)$$

is defined based on L_e, and L_d is always invertible. Since the relations:

$$\begin{bmatrix} x_1(t) \\ e(t) \end{bmatrix} = L_d x(t), \quad \begin{bmatrix} \dot{x}_1(t) \\ \dot{e}(t) \end{bmatrix} = L_d \dot{x}(t) \quad (12)$$

hold, the dynamics of the vector $[x_1^T(t) \ e^T(t)]^T$ for $t \in [t_q, t_{q+1}]$ is given by:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{e}(t) \end{bmatrix} \in (L_d(A_q - (K_q \otimes I_{n_x}))L_d^{-1}) \begin{bmatrix} x_1(t) \\ e(t) \end{bmatrix} + L_d b_q) \oplus \mathcal{W}_{d,q} \quad (13)$$

according to (10) with:

$$\mathcal{W}_{d,q} := \left\{ w \in \mathbb{R}^{n_x n} \middle| -|L_d| w_{m,q} \leq w \leq |L_d| w_{m,q} \right\}.$$

When defining $d(t) := [x_1^T(t) \ e^T(t)]^T$ for simplicity of notation, the initial set $\mathcal{E}(t_0)$ of $d(t_0)$ can be written as:

$$\mathcal{E}(t_0) := \left\{ d \in \mathbb{R}^{n_x n} \middle| |d - \begin{bmatrix} x_{1,0} \\ 0 \\ \vdots \\ 0 \end{bmatrix} | \leq \begin{bmatrix} x_{\max} \mathbf{1}_{n_x} \\ 2x_{\max} \mathbf{1}_{n_x} \\ \vdots \\ 2x_{\max} \mathbf{1}_{n_x} \end{bmatrix} \right\} \quad (14)$$

with $x_{\max} := \frac{\varepsilon}{\sqrt{n_x}}$ according to (5). Starting from $\mathcal{E}(t_0)$, a method to compute a tight over-approximation of the reachable set of $d(t)$ (and thus of the synchronization error $e(t)$) for the time interval $[t_0, t_p]$ is introduced, which is based on the results by (Girard, 2005).

3.3 Reachable Set of the Synchronization Error

As (10) and (13) only hold for the time interval $[t_q, t_{q+1}]$, the reachable set of $d(t)$ is iteratively computed for each $q \in \{0, 1, \dots p-1\}$. As the reachable set of $d(t)$ may be small at the sampling time t_q, while being large otherwise, the reachable set of $d(t)$ is computed for each interval, instead of only at the sampling times.

For the interval $[t_q, t_{q+1}]$, it is known from (13) that the relation:

$$d(t) \in e^{A_{d,q} \delta t_q} \mathcal{E}(t_q) \oplus \int_{t_q}^{t_{q+1}} e^{(t_{q+1} - \tau) A_{d,q}} (L_d b_q \oplus \mathcal{W}_{d,q}) d\tau \quad (15)$$

holds with $A_{d,q} := L_d(A_q - (K_q \otimes I_{n_x}))L_d^{-1}$. The set on the right-hand side of (15) thus contains the true reachable set of $d(t)$ in this interval. To (efficiently) determine this set, the integral $\int_{t_q}^{t_{q+1}} e^{(t_{q+1}-\tau)A_{d,q}}(L_d b_q \oplus \mathcal{W}_{d,q})\,d\tau$ is over-approximated by a hypercube $\mathcal{C}_{l_q} \subseteq \mathbb{R}^{n_x n}$ centered in the origin $\mathbf{0}_{n_x n}$, and all sides have a common length $l_q \in \mathbb{R}^{>0}$ determined by:

$$\begin{aligned} l_q &:= 2\int_{t_q}^{t_{q+1}} e^{(t_{q+1}-\tau)\|A_{d,q}\|_\infty} \| L_d b_q + |L_d| w_{m,q} \|_\infty \, d\tau \\ &:= \frac{2(e^{\delta_{t_q}\|A_{d,q}\|_\infty}-1)}{\| A_{d,q} \|_\infty} \| L_d b_q + |L_d| w_{m,q} \|_\infty . \end{aligned} \quad (16)$$

Let $\mathcal{R}_{[t_q,t_{q+1}]}(\mathcal{E}(t_q))$ denote the true reachable set of $d(t)$ for the interval $[t_q, t_{q+1}]$. The relation:

$$\mathcal{R}_{[t_q,t_{q+1}]} \subseteq \left\{ \bigcup_{t\in[t_q,t_{q+1}]} e^{A_{d,q}t}\mathcal{E}(t_q) \right\} \oplus \mathcal{C}_{l_q} \quad (17)$$

thus applies according to (15) and (16), where $\bigcup_{t\in[t_q,t_{q+1}]} e^{A_{d,q}t}\mathcal{E}(t_q)$ is the union of $e^{A_{d,q}t}\mathcal{E}(t_q)$ for all $t \in [t_q, t_{q+1}]$. Following the method in (Girard, 2005) to determine the union, the reachable set of $d(t)$ at time t_{q+1} is first over-approximated by:

$$\hat{\mathcal{E}}(t_{q+1}) := e^{A_{d,q}t_{q+1}}\mathcal{E}(t_q) \oplus \mathcal{C}_{l_q} \quad (18)$$

according to (17). Based on $\mathcal{E}(t_q)$ and $\hat{\mathcal{E}}(t_{q+1})$, the union $\bigcup_{t\in[t_q,t_{q+1}]} e^{A_{d,q}t}\mathcal{E}(t_q)$ is then over-approximated by a zonotope. The general form of a zonotope is:

$$\mathcal{Z} := \{z \in \mathbb{R}^{n_x n} \mid z = c + \sum_{h=1}^{o} g_h,\ |z| \le 1\} \quad (19)$$

with the center $c \in \mathbb{R}^{n_x n}$ and the generators $g_h \in \mathbb{R}^{n_x n}$, $h \in \{1,\dots,o\}$ of $\mathcal{Z}$. Note that the initial set $\mathcal{E}(t_q)$ in (14) also represents a zonotope $\mathcal{Z}_{t_q}$ with the center $c_{t_q} := [x_{1,q}^T\ \mathbf{0}_{1\times n_x(n-1)}]^T$ and a number of $n_x n$ generators $g_{h,t_q} \in \mathbb{R}^{n_x n}$ with: $[g_{1,t_q},\dots,g_{n_x n,t_q}] := \mathrm{diag}(x_{\max}I_{n_x}, 2x_{\max}I_{n_x},\dots,2x_{\max}I_{n_x})$. Based on $\mathcal{Z}_{t_q}$, a new zonotope $\mathcal{Z}_{[t_q,t_{q+1}]}$ with the center:

$$c_{[t_q,t_{q+1}]} := 0.5(e^{A_{d,q}t_q} + e^{A_{d,q}t_{q+1}})d(t_q) \quad (20)$$

and a number of $2n_x n+1$ generators $g_{h,[t_q,t_{q+1}]} \in \mathbb{R}^{n_x n}$, $h \in \{1,\dots,2n_x n+1\}$ can be determined. The first $n_x n$ generators are obtained to:

$$g_{h,[t_q,t_{q+1}]} := 0.5(e^{A_{d,q}t_q} + e^{A_{d,q}t_{q+1}})g_{h,t_q} \quad (21)$$

for all $h \in \{1,\dots,n_x n\}$, while for the last $n_x n$ generators:

$$g_{h,[t_q,t_{q+1}]} := 0.5(e^{A_{d,q}t_q} - e^{A_{d,q}t_{q+1}})g_{h-(n_x n+1),t_q} \quad (22)$$

applies for all $h \in \{n_x n+2,\dots,2n_x n+1\}$. The remaining $(n_x n+1)$-th generator assumes the value:

$$g_{n_x n+1,[t_q,t_{q+1}]} := 0.5(e^{A_{d,q}t_q} - e^{A_{d,q}t_{q+1}})d(t_q). \quad (23)$$

In this way, the zonotope $\mathcal{Z}_{[t_q,t_{q+1}]}$ is ensured to contain the initial set $\mathcal{E}(t_q)$ and the end set $\hat{\mathcal{E}}(t_{q+1})$. Then, a new hypercube $\mathcal{C}_{\gamma_q}$ is defined with a center at $\mathbf{0}_{n_x n}$ and with all sides having a common length $\gamma_q \in \mathbb{R}^{>0}$:

$$\begin{aligned} \gamma_q := (&e^{t_{q+1}\|A_{d,q}\|_\infty} - t_{q+1} \| A_{d,q} \|_\infty \\ &- 1) \sup_{d\in\{e^{A_{d,q}t_q}\mathcal{E}(t_q)\}} \| d \|_\infty . \end{aligned} \quad (24)$$

In accordance with a result in (Girard, 2005), the relation:

$$\bigcup_{t\in[t_q,t_{q+1}]} e^{A_{d,q}t}\mathcal{E}(t_q) \subseteq \mathcal{Z}_{[t_q,t_{q+1}]} \oplus \mathcal{C}_{\gamma_q} \quad (25)$$

holds. To this end, the reachable set $\mathcal{R}_{[t_q,t_{q+1}]}(\mathcal{E}(t_q))$ is over-approximated by a new set:

$$\hat{\mathcal{R}}_{[t_q,t_{q+1}]}(\mathcal{E}(t_q)) := \mathcal{Z}_{[t_q,t_{q+1}]} \oplus \mathcal{C}_{\gamma_q} \oplus \mathcal{C}_{l_q} \quad (26)$$

according to (17), see also Fig. 1.

Remark 1. *The described procedure to compute the set $\hat{\mathcal{R}}_{[t_q,t_{q+1}]}(\mathcal{E}(t_q))$ is tractable even for a large number of oscillators, which is important for the application to larger networks. This holds true since l_q and γ_q of the two hypercubes can be directly computed according to (16) and (24), while for the zonotope $\mathcal{Z}_{[t_q,t_{q+1}]}$, the number of generators to be computed increases only linearly with the number n of oscillators.*

By iteratively determining the over-approximated reachable sets of Eq. (26), an over-approximated reachable set can be determined for the whole period $[t_0, t_p]$ of the limit cycle consisting of p intervals:

$$\hat{\mathcal{R}}_{[t_0,t_p]}(\mathcal{E}(t_0)) := \bigcup_{q=0}^{p-1} \hat{\mathcal{R}}_{[t_q,t_{q+1}]}(\hat{\mathcal{E}}(t_{q+1})) \quad (27)$$

with $\hat{\mathcal{E}}(t_0) := \mathcal{E}(t_0)$. As only the part $e(t)$ of the vector $d(t) := [x_1^T(t)\ e^T(t)]^T$ refers to the synchronization

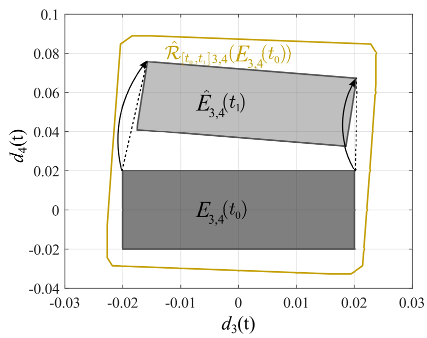


Figure 1: Consider the first reachable set for component three and four of $d(t)$: Starting from $\mathcal{E}_{3,4}(t_0)$, which is also the zonotope $\mathcal{Z}_{t_0,3,4}$, the sets $\hat{\mathcal{E}}_{3,4}(t_1)$ and $\hat{\mathcal{R}}_{[t_0,t_1]3,4}(\mathcal{E}_{3,4}(t_0))$ can be determined, and both are zonotopes.

error, one can project $\hat{\mathcal{R}}_{[t_q,t_{q+1}]}(\hat{\mathcal{E}}(t_{q+1}))$ onto the corresponding subspace in order to obtain the reachable set of $e(t)$. Now let $\sigma := \|\hat{\mathcal{R}}_{[t_0,t_p]}(\mathcal{E}(t_0))\|_{\max}$, such that the sought upper bound of the synchronization error is obtained.

3.4 Determination of Coupling Gains Based on Reachable Sets

While the last section aimed at computing an upper bound on the size of the over-approximated set of synchronization errors, the perspective is now reverted, i.e., given a bounding set for the error, the objective is to obtain admissible time-varying coupling gains.

First of all, a possibly time-varying bounding set $\mathcal{E}_{\max}(t) \subseteq \mathbb{R}^{n_x n}$ of the maximally allowable synchronization errors is chosen for any $x_j(t) - x_1(t)$, $j \in \{2,\ldots,n\}$. Corresponding to Assumption 2, it makes sense to choose again a piecewise constant bounding set $\mathcal{E}_{\max}(t) = \mathcal{E}_{\max,q}$ for $t \in [t_q, t_{q+1}]$. The objective is now to synthesize the coupling gains $k_{ij,q}$ such that the reachable set of $x_i(t) - x_1(t)$ within one period never leaves $\mathcal{E}_{\max,q}$. An option to obtain $k_{ij,q}$ is to start from an initial value and then to iterate it until the size of the over-approximation of the set of reachable synchronization errors is limited to $\mathcal{E}_{\max,q}$.

To do so, first determine the period T of the limit cycle of (3) and set t_p equal to T while $t_0 = 0$. The period is divided into p intervals, whereby it should be noted that although the $k_{ij,q}$ are assumed to be different for each time interval $[t_q, t_{q+1}]$, the coupling gains can also be constant for a series of time intervals $[t_q, t_{q+1}]$, …, $[t_{q+v}, t_{q+1+v}]$ with $v \in \mathbb{N}, v < p$.

Starting from initial values $x_{i,0}$ close to a point of the limit cycle of the averaged dynamics, an initial set $\mathcal{E}(0)$ results as in (14), and the reachable set $\hat{\mathcal{R}}_{[0,T]}(\mathcal{E}(0))$ can thus be determined. By allowing the coupling gains to be adjusted in each interval $[t_q, t_{q+1+r}]$, the reachable set $\hat{\mathcal{R}}_{[t_{q+l},t_{q+1+l}]}(\hat{\mathcal{E}}(t_{p+l}))$, $l \in \{0,\ldots,v+1\}$ can be successively constructed. The synthesis task, accordingly, is to find suitable gains $k_{ij,q}$ for each interval (e.g. the smallest ones which are just sufficient to keep the error within the selected bound).

Theorem 1. *Let the projections of $\hat{\mathcal{R}}_{[t_{q+l},t_{q+1+l}]}(\hat{\mathcal{E}}(t_{p+l}))$, $l \in \{0,\ldots,v+1\}$ onto each subspace of $e_{(j,1)}(t) := x_j(t) - x_1(t)$, $j \in \{2,\ldots,n\}$ be denoted by $\hat{\mathcal{R}}_{[t_{q+l},t_{q+1+l}]r,r+1}(\mathcal{E}_{r,r+1}(t_0))$ with the dimensions $r \in \{1,\ldots,n_x n-1\}$. If:*

a.) the projected sets $\hat{\mathcal{R}}_{[t_{q+l},t_{q+1+l}]r,r+1}(\mathcal{E}_{r,r+1}(t_0))$ with the dimensions $r \in \{1,\ldots,n_x n-1\}$ are contained in $\mathcal{E}_{\max,q}$ for all coordinates r and all q, and if

b.) $\hat{\mathcal{E}}(T)$ is contained in the initial set $\mathcal{E}(t_0)$ at the end of the last interval,

then the synchronization error $e_{(j,1)}(t)$, $j \in \{2,\ldots,n\}$, never exceeds $\mathcal{E}_{\max,q}$ for the whole phase of preserving synchronization.

Proof. Starting from the first interval of the limit cycle, the if-condition under a.) ensures that the synchronization error starting from $\mathcal{E}(0)$ at $t = 0$ is always bounded by the $\mathcal{E}_{\max,q}$ within the first period. At the time $t = T$, which is the end of the first period as well as the beginning of the second period, the second if-condition b.) ensures that the initial set $\hat{\mathcal{E}}(T)$ of the second period is a subset of $\mathcal{E}(0)$. Thus, the synchronization error within the second period is further bounded by the $\mathcal{E}_{\max,q}$ and contained in $\mathcal{E}(0)$ at the end. Recursive application guarantees that the error is bounded by the $\mathcal{E}_{\max,q}$ for all $t > 0$. □

By using this scheme, Problem 2 of synthesizing the coupling gains $k_{ij,q}$ is solved. Note that the set $\hat{\mathcal{R}}_{[t_{q+l},t_{q+1+l}]}(\hat{\mathcal{E}}(t_{p+l}))$, $l \in \{0,\ldots,v+1\}$ for each interval depends explicitly on the linearization (10) and the Laplacian matrix L.

4 NUMERICAL EXAMPLES

In the following, two settings with two different classes of nonlinear limit cycle oscillators are considered. For the first setting, a constant gain $k_{ij}(t) = k$ is used, while a time-variable $k(t)$ is synthesized for the respective oscillator network in the second setting. For the sake of simplicity, the gains for all coupling links are selected to be the same, a uniform step size $\delta t_q = \delta t > 0$ is selected, and $\mathcal{E}_{\max}(t)$ is chosen to be constantly $\mathcal{E}_{\max}$ for all times.

4.1 Coupled van-der-Pol Oscillators

In the first test, the synchronization problem for an example of $n = 10$ nonidentical van-der-Pol oscillators:

$$\begin{bmatrix} \dot{x}_{i,1}(t) \\ \dot{x}_{i,2}(t) \end{bmatrix} = \begin{bmatrix} x_{i,2}(t) \\ -x_{i,1}(t) + \mu_i(1 - x_{i,1}(t)^2)x_{i,2}(t) \end{bmatrix} \tag{28}$$

with significantly different parameter $\mu_i > 0$ and $i \in \mathcal{N}$ is considered, see e.g. (Trummel et al., 2023). The value of μ_i decides the form of each local limit cycle.

It can be noticed from their limit cycles in Fig. 2 that even without coupling, the limit cycles are close to each other in the region marked as D, while far to each in the region G. Hence, one can infer that a

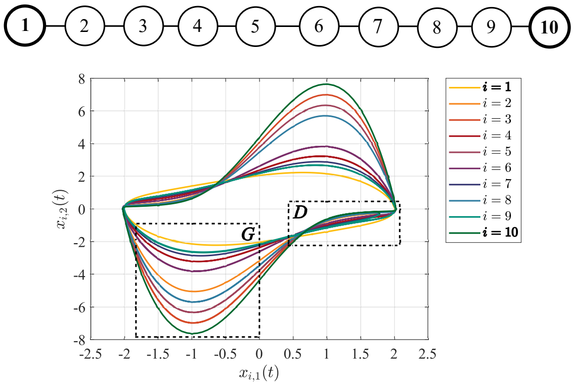


Figure 2: The considered 10 oscillators are assumed to be connected by undirected edges in chain structure, while their parameters are selected to: $\mu_1 = 0.5, \mu_2 = 3, \mu_3 = 4.5, \mu_4 = 1.5, \mu_5 = 4, \mu_6 = 2, \mu_7 = 1.2, \mu_8 = 3.5, \mu_9 = 1, \mu_{10} = 5$. Their uncoupled limit cycles are quite different in the region G, while being quite similar in the region D.

larger coupling gain would be required to preserve the synchronization in G than in D. To examine this, the reachable set $\hat{\mathcal{R}}_{[0,t_p]}(\mathcal{E}(t_0))$ of the synchronization error is computed for using a gain $k = 20$ – once for an initial set in the region G (on the limit cycle of the averaged dynamics (3)), and once with an initial set in D, see Fig. 4. Note that the obtained reachable set $\hat{\mathcal{R}}_{[0,t_p]}(\mathcal{E}(t_0))$ is projected onto the subspace of the error $e_{10,1}(t)$ for illustration reasons. This pair of oscillators is chosen since the parameters $\mu_1 = 0.5$ and $\mu_{10} = 5$ differ the most over all oscillators. By comparing the reachable sets in Fig. 4b and Fig. 4d, one can notice that the reachable set starting from the region D in the former plot is much smaller than the one starting from G even for a larger time interval ($t_p = 1.5$ versus $t_p = 0.75$). This shows that, in order to keep the synchronization error below a certain bound, a coupling gain smaller than $k = 20$ can be applied for the region D, while a larger one is required for the region G.

Then, the coupling gain k for two diffusively coupled van-der-Pol oscillators with $\mu_1 = 0.5$ and $\mu_2 = 2$ is synthesized based on the reachable set (see the limit cycles of the two oscillators and the one of the averaged dynamics (3) in Fig. 6a). A given set $\mathcal{E}_{\max}$ of a maximally permitted synchronization error is shown in Fig. 6c. It is shown that by using the time-varying coupling gain in Fig. 6e, the reachable sets of the synchronization error are always contained in $\mathcal{E}_{\max}$. At the same time, the reachable set $\hat{\mathcal{E}}(T)$ at the end of a period with $T = 6.8$ and $v = 34$ is also contained in the initial set $\mathcal{E}(0)$, see Fig. 6d.

4.2 Coupled Merkin–Needham–Scott Oscillators

In the second test, a number of six diffusively coupled Merkin–Needham–Scott oscillators (Saha and Gangopadhyay, 2017) in the form of:

$$\begin{bmatrix} \dot{x}_{i,1}(t) \\ \dot{x}_{i,2}(t) \end{bmatrix} = \begin{bmatrix} -x_{i,1}(t) + (\eta_i + x_{i,1}(t)^2)x_{i,2}(t) \\ \beta_i - (\eta_i + x_{i,1}(t)^2)x_{i,2}(t) \end{bmatrix}$$

with nonidentical parameters $\eta_i \geq 0$, $\beta_i \in \mathbb{R}$, and $i \in \mathcal{N}$ are considered, see Fig. 3. The regions D and G are defined equivalently to the previous example, and again the oscillators with the largest difference between their limit cycles are taken into account. It can be noticed that the reachable error sets starting from region D are much smaller than the one initializing from area G. Therefore, a gain much smaller than 5 can be applied to obtain a synchronization which is similar to the one for $k = 5$. For the area G, the reverse case is observed: the gain has to be increased to reduce the size of the reachable sets. This observation can once more be examined by computing the reachable set of the synchronization error for using a gain $k = 5$, and similar as the first test, once starting from the region G and once from the region D. The resulting reachable sets are demonstrated in Fig. 5, which confirmed the observation.

Finally, the coupling gains for two coupled Merkin–Needham–Scott oscillators (see Fig. 7a) with parameters $\eta_1 = 0.1$, $\beta_1 = 0.6$, $\eta_2 = 0.05$, and $\beta_2 = 0.5$ are synthesized for the phase of preserving synchronization. The application of the procedure explained in Sec. 3.4 results in the time-varying coupling gains shown in Fig. 7e, and satisfying $\mathcal{E}_{\max}$ as in Fig. 7c. Obviously, $\mathcal{E}_{\max}$ contains $\hat{\mathcal{E}}(T)$ at the end of a period with $T = 10.05$ and $v = 201$, see Fig. 7d.

Thus, the time-varying coupling gain satisfies the two conditions in Sec. 3.4, making it a good choice for preserving synchronization.

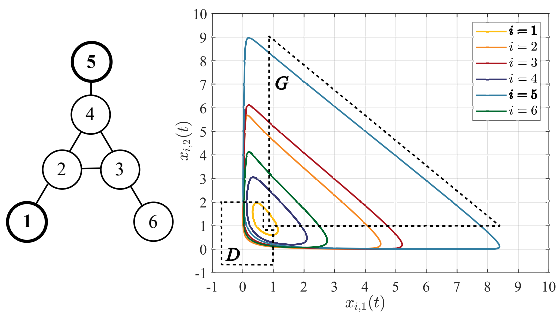
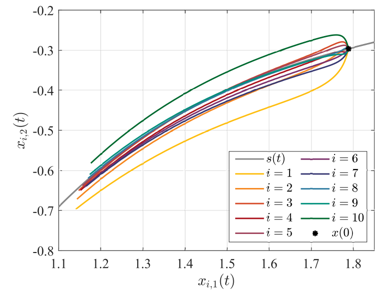
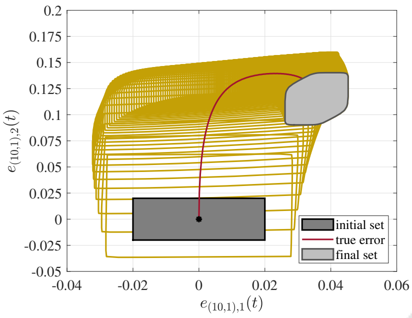


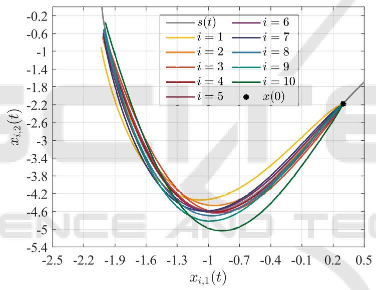
Figure 3: Network of 6 Merkin–Needham–Scott limit cycle oscillators with parameters: $\eta_1 = 0.1, \eta_2 = 0.01, \eta_3 = 0.01, \eta_4 = 0.05, \eta_5 = 0.005, \eta_6 = 0.02$, and $\beta_1 = 0.6, \beta_2 = 0.2, \beta_3 = 0.3, \beta_4 = 0.5, \beta_5 = 0.4, \beta_6 = 0.25$. Similar to the van-der-Pol oscillator example, the trajectories in region D are more similar than the ones in area G.



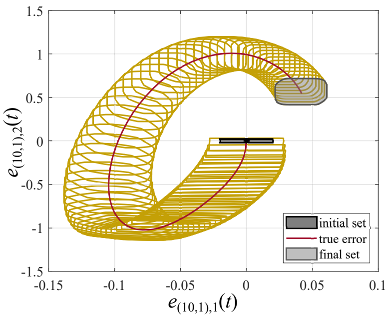
(a)



(b)

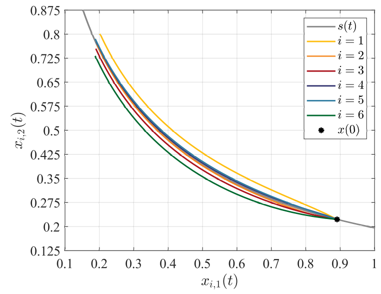


(c)

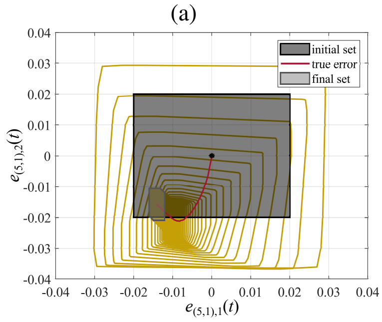


(d)

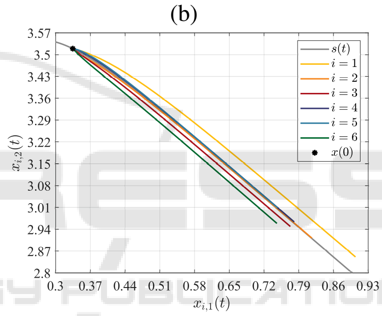
Figure 4: By using a coupling gain $k=20$, the reachable set of $e_{(10,1)}(t)$ is computed once starting from the region D and once from the region G in Fig. 2. The gray limit cycle segments in (a) and (c) are the one from the averaged dynamics (3). The zonotopes in yellow in (b) and (d) are the intermediate reachable sets $\hat{\mathcal{R}}_{[t_q,t_{q+1}]}(\hat{\mathcal{E}}(t_{q+1}))$ projected onto the space of $e_{(10,1)}(t)$ with $\delta t = 0.005$. The union of those intermediate reachable sets is thus the reachable set of the whole interval $[0,1.5]$ for (b) and $[0,0.75]$ for (d).



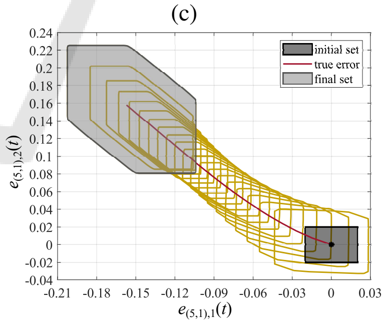
(a)



(b)



(c)



(d)

Figure 5: Reachable sets of $e_{(5,1)}(t)$ determined for $k=5$, once starting from the region D and once from G in Fig. 3. The parts (a) and (c) show the gray limit cycle segments from the averaged dynamics (3). The yellow colored zonotopes in (b) and (d) are the intermediate reachable sets $\hat{\mathcal{R}}_{[t_q,t_{q+1}]}(\hat{\mathcal{E}}(t_{q+1}))$ projected onto the space of $e_{(5,1)}(t)$ with $\delta t = 0.05$. The union of these reachable sets is the reachable set for the whole interval $[0,2]$ for (b) and $[0,1]$ for (d).

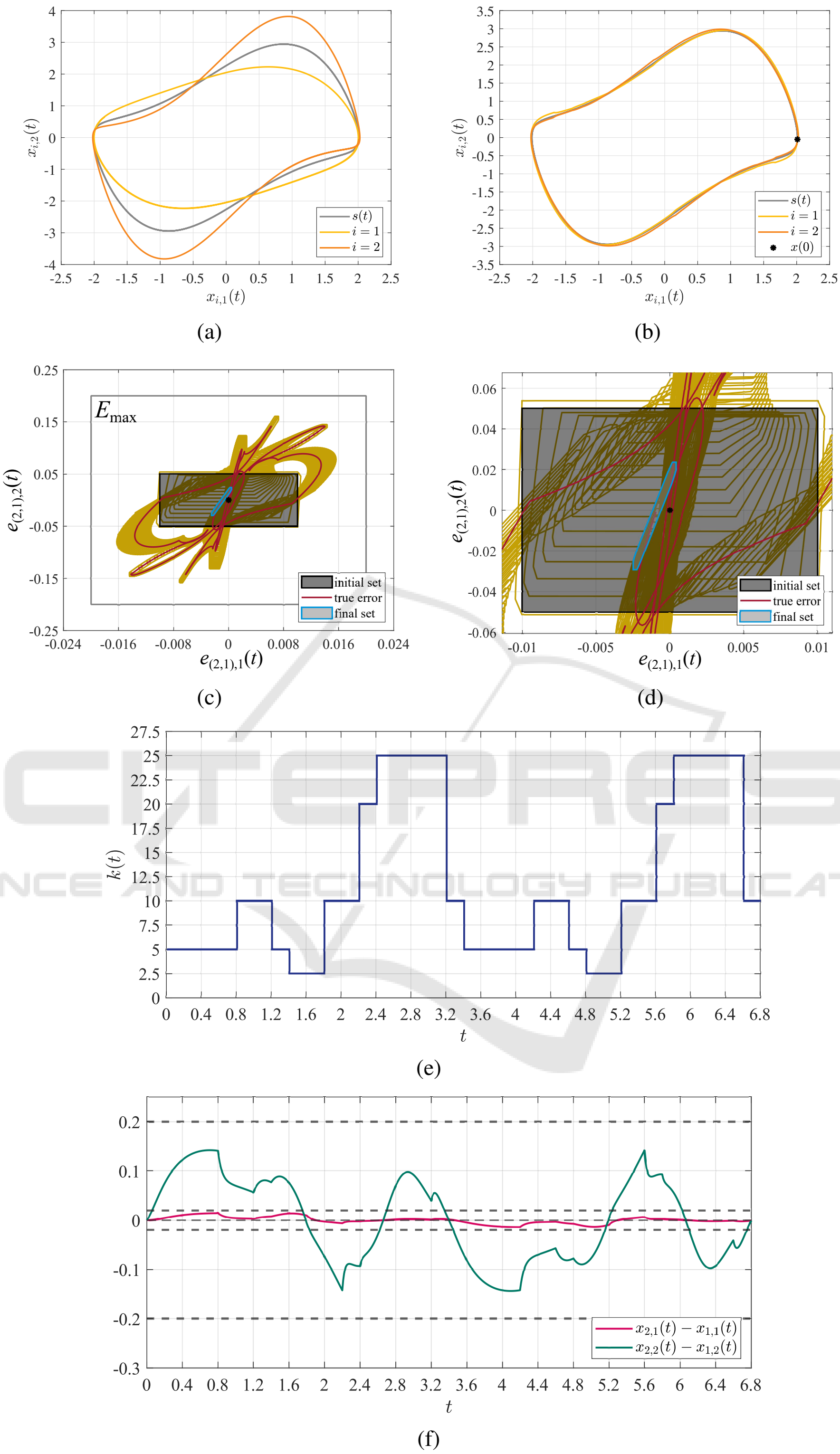


Figure 6: By using the time-varying coupling gains in (e), the reachable sets of the synchronization error in (c) are always contained in $\mathcal{E}_{\max}$ for the whole period, while $\hat{\mathcal{E}}(T)$ at the end of a period is also contained in $\mathcal{E}(0)$, see (d). A possible evolution of the synchronization error $e_{(2,1)}(t)$ in (f) of the synchronizing states in (b) confirms the result.

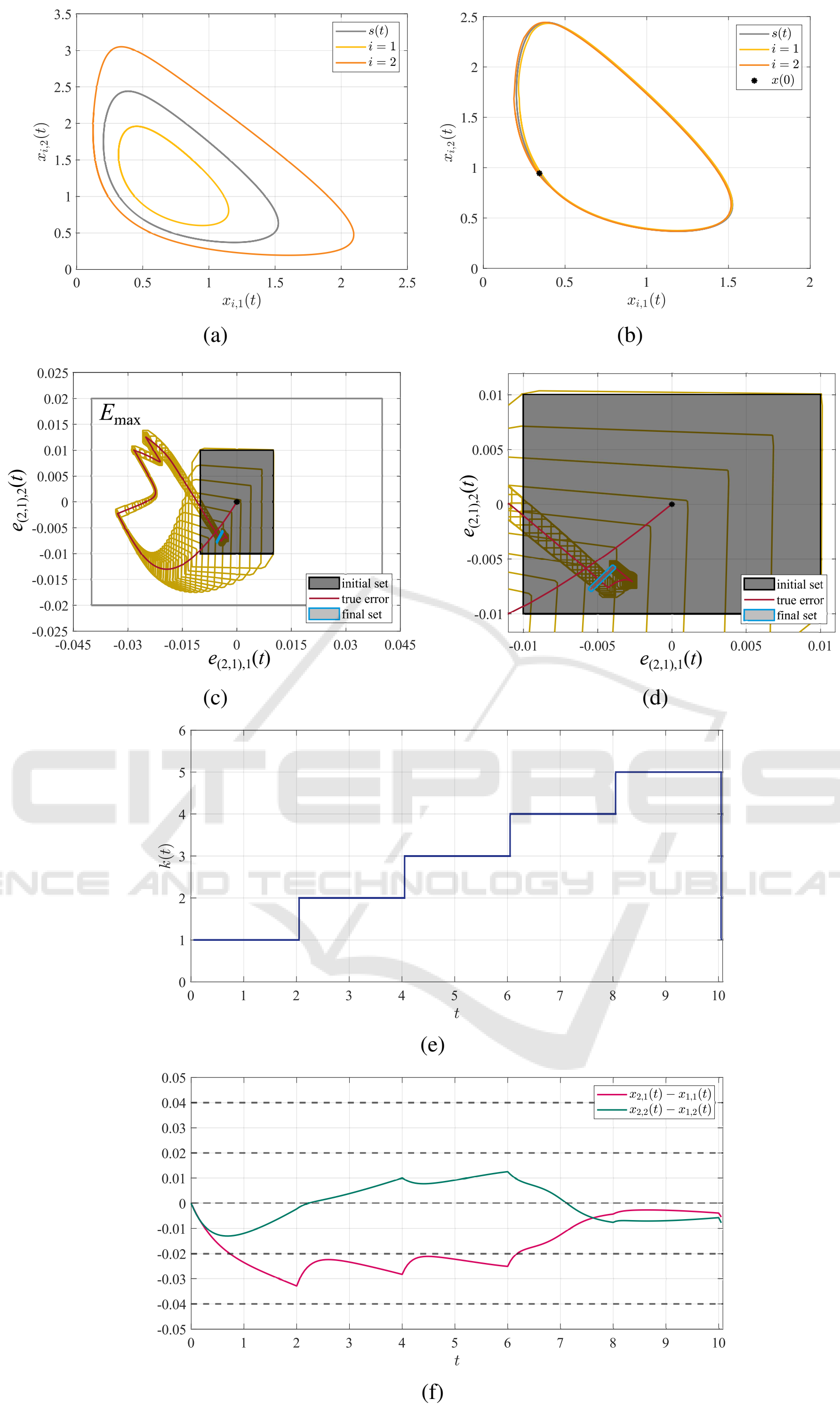


Figure 7: The application of the time-varying coupling gains in (e) leads to the reachable sets of the synchronization error in (c), while they are always contained in $\mathcal{E}_{\max}$ for the whole period. $\hat{\mathcal{E}}(T)$ at the end of a period is also contained in $\mathcal{E}(0)$, see (d). In (f), the evolution of the synchronization error $e_{(2,1)}(t)$ is illustrated for the synchronizing states in (b).

5 CONCLUSIONS

The paper has provided techniques to analyze and synthesize bounds of the synchronization error for a network of diffusively coupled nonidentical nonlinear limit cycle oscillators. Since the synchronization error cannot be derived as analytic expression due to the nonlinear dynamics of the local oscillators, over-approximating reachable sets of the error are determined for evaluation over time. Based on the obtained reachable set, a suitable coupling gain to preserve the synchronization can be synthesized with a guaranteed bound on the maximal synchronization error. Effectiveness of this method is confirmed in different simulations with respect to both, the reachable set computations and the synthesis of the coupling gain. The methods can be applied to any possible coupling topology of the oscillator network.

Future work aims at applying the method to other types of nonlinear oscillators, and to the consideration of exogenous signals imposed on the oscillators.

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