A Whole New World: Can Virtual Reality Help to Understand Non-Euclidean Geometries?

Maé Mavromatis[1][a], Ronan Gaugne[2][b], Rémi Coulon[3][c] and Valérie Gouranton[1][d]
[1]*Université Rennes, INSA Rennes, Inria, CNRS, IRISA, France*
[2]*Université Rennes, Inria, CNRS, IRISA, France*
[3]*Université de Bourgogne, CNRS, IMB, France*
mae.mavromatis@inria.fr, {ronan.gaugne, valerie.gouranton}@irisa.fr, remi.coulon@cnrs.fr

Keywords: Virtual Reality, Education, Simulation, Non-Euclidean Geometry, User Study.

Abstract: With the democratisation of digital technologies, new pedagogical approaches are emerging that leverage these innovative media to enhance student engagement and promote different ways of learning. This article compares three learning modalities—slides, screen, and VR—in terms of knowledge acquired, time spent, and usability. The *slides* modality involves an illustrated slide presentation, the *screen* modality uses an on-screen simulation with navigation, and the VR modality shows the same simulation in virtual reality with a Head-Mounted Display (HMD). In this study, we investigated the impact of these modalities on students' understanding of the essential properties of the unintuitive non-Euclidean geometries $\mathbb{S}^3$ and $\mathbb{H}^3$. All three modalities helped participants improve their answers to the mathematics questionnaire, though further research is needed to fully exploit the unique benefits of virtual reality.

1 INTRODUCTION

Virtual reality (VR) is advancing rapidly, with significant progress in fields like medicine, entertainment, and education. As VR becomes more accessible, many studies focus on its potential to enhance student learning (Mikropoulos and Natsis, 2011). New educational opportunities are emerging, widely recognized as beneficial. The use of VR in education is expected to grow, particularly in mathematics, where it improves motivation and performance (Lai and Cheong, 2022). VR immerses students in a virtual world, enhancing their mathematical reasoning and spatial skills (Kaufmann and Schmalstieg, 2003).

A key advantage of VR is its ability to represent 3D objects in a 3D environment, aiding 3D thinking and mental transformation, which 2D technologies cannot provide (Hedburg and Alexander, 1994). This is especially useful in geometry, particularly non-Euclidean geometries, which are counter-intuitive and difficult to understand due to their conflict with classical geometry. Several studies have explored innovative ways to teach these geometries (Sukestiyarno et al., 2023).

However, few studies examine VR's role in teaching non-Euclidean geometries. This work aims to assess the impact of VR on learning non-Euclidean geometries, comparing it to screen simulations and traditional slide-based explanations, to determine if immersion helps students understand these abstract spaces.

In a user study, we created three learning environments for non-Euclidean geometries, tested with participants from diverse backgrounds. The key mathematical properties of each geometry, requiring various skills, are presented in section 3. Understanding of the geometries was assessed before and after the experiment to measure the effectiveness of these environments and compare them. The experiment in section 5 shows participants improved their answers to the mathematics questionnaire, with no statistically significant difference between the three modalities.

[a] https://orcid.org/0000-0001-9089-7859
[b] https://orcid.org/0000-0002-4762-4342
[c] https://orcid.org/0000-0003-0233-5974
[d] https://orcid.org/0000-0002-9351-2747

Mavromatis, M., Gaugne, R., Coulon, R. and Gouranton, V.
A Whole New World: Can Virtual Reality Help to Understand Non-Euclidean Geometries?.
DOI: 10.5220/0013150900003912

In *Proceedings of the 20th International Joint Conference on Computer Vision, Imaging and Computer Graphics Theory and Applications (VISIGRAPP 2025) - Volume 1: GRAPP, HUCAPP and IVAPP*, pages 231-238
ISBN: 978-989-758-728-3; ISSN: 2184-4321

2 RELATED WORKS

2.1 VR in Mathematics Education

Virtual reality has made significant progress in education, with current research focusing on its use to teach real-world phenomena and provide immersive learning experiences for students (Lai and Cheong, 2022). Studies suggest that students are interested in using virtual reality as part of their courses (Baxter and Hainey, 2019).

Using virtual reality as a teaching aid in mathematical geometry is an innovative approach. Immersive technologies like VR offer many advantages (Cevikbas et al., 2023), such as increasing motivation, providing a different learning experience, and enhancing effectiveness (Osypova et al., 2021). Virtual reality can help students grasp complex concepts and reduce misunderstandings (Mikropoulos and Natsis, 2011). However, some studies show mixed results. For example, a study on virtual reality for multivariable calculus found that students sometimes performed worse on certain questions after using the VR application, though results varied. Measuring these effects is challenging, as comparing VR environments with traditional classrooms is difficult (Kang et al., 2020).

Virtual reality learning aids have key benefits, such as boosting motivation, visualizing abstract concepts, and stimulating interest in new knowledge (Wang et al., 2018). VR also allows students to play virtual characters, manipulate geometries, and learn geometry concepts (Guerrero Idrovo et al., 2016).

One of the key advantages of VR over other technologies is its ability to represent three-dimensional (3D) objects in a 3D world, which is particularly useful in geometry.

2.2 VR Geometry Teaching Platforms

Virtual reality mathematical geometry teaching platforms such as ClassVR, VRMath and GeoGebra are available on the market today. The ClassVR platform provides an operating record system that allows teachers to understand how students learn using the platform. The VRMath platform provides virtual geometric mathematical objects, such as cylinders, cones and trigonometric cones. GeoGebra is a three-dimensional drawing platform that provides a learning space in which students can add points and lines in virtual space to create a 3D virtual object. Research continues to use virtual reality to develop immersive learning systems for geometric mathematics (Su et al., 2022). Recently, the MathworldVR virtual reality (WebVR) application has been proposed for teaching higher mathematics concepts that require spatial abilities. It allows, for example, the manipulation of input variables of parametric functions, which reduces the time needed by students to understand the underlying principles of a given mathematical theory (Takac, 2020).

2.3 Non-Eucliean Geometry Education

Over the past few decades, there has been increasing recognition that mathematics is considered a difficult subject, requiring changes in teaching methods and tools to boost students' interest and participation (Gambini and Lénárt, 2021). Non-Euclidean geometry, in particular, is abstract and challenging to learn. Multiple studies have used different approaches to support students in their learning process. For example, one study assessed the impact of an ethnomathematics approach on students' spatial abilities for Euclid's, Lobachevsky's, and Riemann's geometries, finding a positive influence on spatial skills (Sukestiyarno et al., 2023). Another explored teaching non-Euclidean concepts using astronomical images, where students learned to calculate distances and areas on the Moon's surface, aiding their understanding of geodesics and spherical triangles (Caerols et al., 2021). An experiment compared Euclidean and non-Euclidean geometries to help students' understand geometric properties and challenge their perceptions of mathematics (Gambini, 2021). Other research has focused on integrating non-Euclidean geometry into high school curricula to broaden perspectives and clarify concepts like undefined terms in Euclidean geometry (Buda, 2017).

The use of technology has created new dynamics in teaching geometry, enhancing students' understanding. For instance, a study involving technology in teaching hyperbolic geometry through Poincaré's Disk found it increased student engagement and helped them understand key concepts of this geometry (Kotarinou and Stathopoulou, 2017).

2.4 Deriving the Approach

The value of virtual reality in education is being increasingly studied, yet its potential use in non-Euclidean geometry remains unexplored. This area of mathematics is complex and unintuitive, and could benefit from new methods of visualization and understanding. In this study, we focused on designing the pedagogical content of the different experimental conditions and examining the user experience (cybersickness, acceptability, usability, and behavior).

3 UNDERLYING MATHEMATICAL CONCEPTS

Euclidean geometry is well-known, but alternative 2D geometries, such as the sphere and hyperboloid (see Fig. 1), also exist. In three dimensions, Thurston's geometrization theorem (Thurston, 1986), proven by Perelman (Eres et al., 2010), classifies all possible geometries into eight distinct models. These include the familiar 3D Euclidean space, along with seven non-Euclidean geometries, such as spherical geometry $\mathbb{S}^3$ and hyperbolic geometry $\mathbb{H}^3$. *The sphere* $\mathbb{S}^2$ *and hyperboloid* $\mathbb{H}^2$ *are used throughout the article to illustrate their specificities.*

3.1 $\mathbb{S}^3$ and $\mathbb{H}^3$

$\mathbb{S}^3$ (resp. $\mathbb{H}^3$) is the 3D analogue of the 2D sphere (resp. hyperbolic plane) in $\mathbb{R}^3$. We model $\mathbb{S}^3$ as the unit sphere in $\mathbb{R}^4$ and $\mathbb{H}^3$ as the Minkowski hyperboloid. The distance between two points P_1 and P_2 is the length of the shortest curve joining them. Both $\mathbb{S}^3$ and $\mathbb{H}^3$ are homogeneous and isotropic (all points, resp. directions, are equivalent).

3.2 Mathematical Notions and Related Properties

Geodesics: A geodesic (see Fig. 1) in a space S is a curve in S that locally minimises the distance travelled. It is the analogue of a straight line in a plane.

Holonomy: The holonomy of a connection measures how parallel transport along closed loops alters the geometric information. This change results from the connection's curvature. For example, Fig. 2 shows the orientation changes of an observer traveling along the sides of a triangle on a 2D sphere.

Conjugate Points: Conjugate points indicate when geodesics fail to minimise length globally. For example, on a sphere, geodesics passing through the North Pole can be extended to the South Pole, meaning segments containing both poles do not minimise length. Any pair of antipodal points on the 2D sphere are conjugate points.

Objects Visible in Front of and Behind: The geodesics of the sphere are great circles centered at the sphere's center (see Fig. 1). As a result, an object in front of an observer on $\mathbb{S}^3$ is also visible behind, as a light ray can travel along the geodesic passing through the object in both directions and return to the observer.

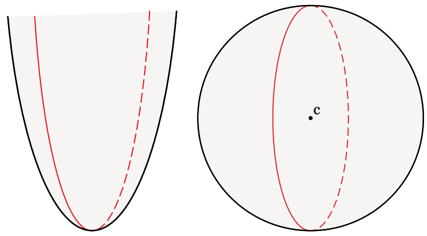

Figure 1: Left, an hyperboloid in 2D. Right, a sphere in 2D. The analogues to $\mathbb{H}^3$ and $\mathbb{S}^3$ in 2D. In red, a geodesic.

Distance Between Objects: The shape of geodesics varies across geometries, and even within a single geometry, affecting how distances are perceived. In particular, size is not always an indicator of distance (see Fig. 3).

Right Angled Regular Polygons: Due to space curvature, the properties of polygons vary across geometries. For example, the triangle obtained by cutting a sphere in eight similar pieces is a right angled equilateral triangle (see Fig. 2), while in Euclidean geometry, the sum of a triangle's angles is always 180°. In $\mathbb{H}^3$, one can construct any n-gone with $n \geq 5$ as a right angled regular polygon.

4 METHOD

This work aims to design and evaluate a VR-based learning environment for non-Euclidean geometries. This section covers the simulation method, the design focusing on key properties of the geometries, and the evaluation approach.

4.1 Simulating Non-Euclidean Spaces

Common rendering software does not support non-Euclidean geometries, as light rays follow straight lines in traditional engines. Based on the method in (Coulon et al., 2022), we implemented a raymarching renderer that traces rays along the geodesics of the geometry. Other renderers for Thurston geometries include (Weeks, ; Berger, 2015; Kopczyński et al., 2017; Velho et al., 2020). Illumination is computed using Phong's model.

4.2 Learning Environment Design

During the experiment's design, we aimed to identify key properties of the chosen geometries, focusing

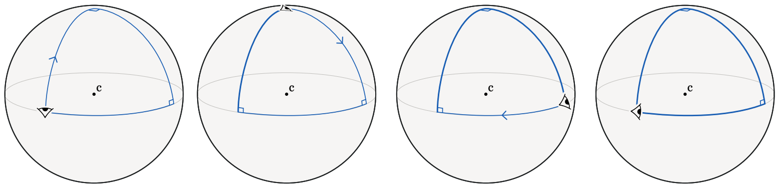

Figure 2: Illustration of the principle of holonomy on a 2-dimensional sphere. The observer moves along a right angles triangle without ever turning. Their orientation changes as they move along the triangle. When they have completed a full turn of the triangle, they have completed a quarter turn in terms of orientation.

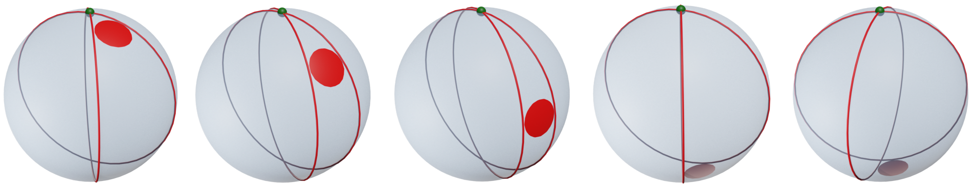

Figure 3: This series of figures illustrates the proportion of an observer's field of view occupied by an object as a function of its distance from the observer. The object here is the red disc, while the observer is the green dot. The two red geodesics tangent to the disc form an angle corresponding to the size of the disc as perceived by the observer. This angle decreases and then increases as the disc moves from the observer's position to the antipodal position.

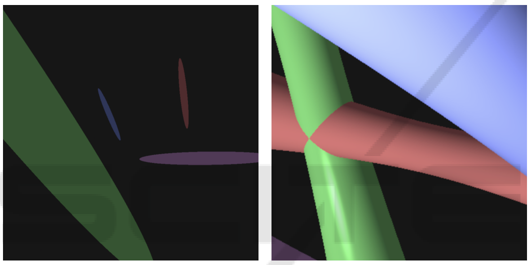

Figure 4: Screenshot of the virtual environment featuring four infinite cylinders placed around geodesics in $\mathbb{H}^3$ (left) and $\mathbb{S}^3$ (right).

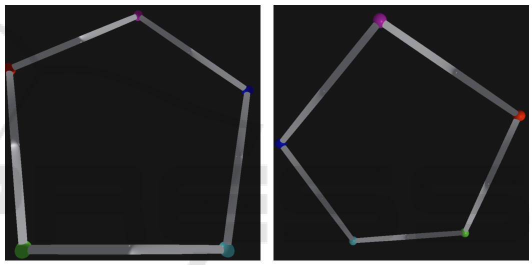

Figure 5: Screenshots of the right angled pentagon in $\mathbb{H}^3$ from two different points of view.

on both simple properties like distance and geodesics, and more complex ones like holonomy. We also considered the logical order for presenting these properties, as some concepts build on others. Presenting them randomly would have been irrelevant. This section presents these properties in order and explains how they were introduced to the participants.

Geodesics: The aim was to enable participants to identify the shape of geodesics in $\mathbb{S}^3$ and $\mathbb{H}^3$. The virtual environment included four infinite cylinders positioned around four geodesics (see Figure 4).

Objects Visible in Front of and Behind: This property only concerned $\mathbb{S}^3$. The aim was for participants to see that objects visible in front of them were also visible behind them. The virtual environment included a finite cylinder and a sphere.

Distance: The aim was for participants to understand that distance estimation differs between geometries. To illustrate this, spheres in the virtual environment were coloured based on their distance from the user: light for close spheres and dark for distant ones. The environment consisted of two spheres and finite cylinders regularly positioned in space.

Conjugate Points: This property applied only to $\mathbb{S}^3$ and aimed to help participants understand conjugate points. The virtual environment featured a finite cylinder and a sphere, both color-coded by distance from the user. In the VR and *screen* conditions, participants were asked to identify the position where they would feel enclosed by one of the objects.

Right Angled Regular Polygons: The goal was to demonstrate that polygonal properties vary across geometries. Participants were shown regular polygons with right angles: an equilateral triangle with right angles in $\mathbb{S}^3$ (see Figure 2) and a regular pentagon with right angles in $\mathbb{H}^3$ (see Figure 5). We explained on the UIs and slides that polygons with more than five sides can be constructed as right-angled regular poly-

Figure 6: The user is immersed in a tiled hyperbolic space made up of sphere complements. This tiling reveals pillars that highlight the rotations in the environment caused by holonomy.

gons in $\mathbb{H}^3$. In the VR and *screen* conditions, participants moved around to observe the angles, while in the *slides* condition, they viewed a drawing.

Holonomy: The aim was to help participants understand the effect of holonomy for a geometry dweller. The virtual environment featured a tiled space where "pillars" appeared (see Figure 6). In the VR condition, participants made squares with their head and observed the environment. In the *screen* condition, they used the keyboard to move the camera and observe the environment. In the *slides* condition, a series of drawings illustrated how the observer's orientation changed as they walked along the right-angled polygon from the previous slide (see Figure 2).

5 EXPERIMENT

The aim of this experiment was to explore the impact of three technologies (VR, *screen*, *slides*) on understanding non-Euclidean geometries. In the VR and *screen* conditions, participants navigated a virtual environment and focused on one property of the geometry per scene. In the *slides* condition, participants worked through one slide per property. They were free to spend as much time as needed on each property, exploring the environment in the VR and *screen* conditions, and reading the slides in the *slides* condition. Participants were divided into two groups: *experts* (mathematics students at the Master's or PhD level) and *novices* (students in preparatory classes or computer scientists).

5.1 Participants and Apparatus

19 *experts* participants took part in the experiment (5 females, 12 males, 1 other, 1 preferred not to answer), aged 19 to 28 ($\overline{X} = 22, SD = 2.5$) and 21 *novices* participants took part in the experiment (3 females, 16 males, 2 other), aged 19 to 60 ($\overline{X} = 26, SD = 8.8$).

Figure 7: From left to right: Slides, Screen, and VR setups.

All participants were students recruited on our campus or computer scientists from the lab. They did not receive any financial compensation. All had normal or corrected vision. Participants were divided into three groups for the three conditions (see Figure 7), with the groups balanced according to their registration order. Those in the VR condition were immersed in the virtual environment using an Oculus HMD and associated controllers. The VR experiment was conducted using a desktop computer ensuring a minimum of 80 *fps* under all conditions, developed using Unity 2023.2 The experiment was approved by the local ethics committee (COERLE 2024-37).

5.2 Experimental Protocol and Design

After signing the consent form, participants completed a demographic questionnaire and were briefed on the experiment's purpose. They were given a document explaining the mathematical concepts they would encounter; the expert version was technical, while the novice version was more general. Participants then took a math questionnaire to assess their baseline understanding of the first geometry. Depending on their assigned condition, participants were equipped with the appropriate setup: VR equipment for the *vr* condition, or a screen, mouse, and keyboard for the *screen* or *slides* conditions. The experiment consisted of two blocks, each corresponding to one geometry ($\mathbb{S}^3$ or $\mathbb{H}^3$). Each block had scenes addressing different properties (6 scenes for $\mathbb{S}^3$, 4 for $\mathbb{H}^3$). Participants in the VR or *screen* conditions filled out a sickness questionnaire after each scene (i.e. each property). After completing the first block, participants re-took the math questionnaire for the first geometry and then completed it for the second geometry. The same procedure was followed for the second block, with the sickness questionnaire completed after each property. After finishing the second block, participants filled out the math questionnaire for the second geometry and completed an acceptance questionnaire.

The order of geometries was counterbalanced. Participants in the VR or *screen* conditions spent around 20 minutes per block, while the *slides* condition took approximately 45 minutes in total.

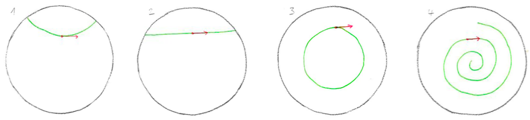

Figure 8: Drawings presented to participants when answering the maths questionnaire about $\mathbb{H}^3$.

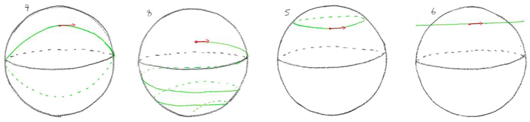

Figure 9: Drawings presented to participants when answering the maths questionnaire about $\mathbb{S}^3$.

5.3 Experimental Data

Subjective Measures. For the VR and *screen* conditions, we used the VRSQ questionnaire (Kim et al., 2018) to assess the participants' VR sickness level. After the experiment, participants in all three conditions completed a subset of the UTAUT2 questionnaire with 7-point Likert scale answers (Venkatesh et al., 2012) to assess acceptance. We excluded the UTAUT2 sections "social influence", "habit" and "price value", as they were irrelevant to our study.

Objective Measures. We measured the time spent on each scene or slide based on the condition, as well as the displacement in the virtual environment for participants in the VR and *screen* conditions.

Before and after each block, participants completed a custom math questionnaire, created in collaboration with a mathematician. The questionnaire included six questions, one for each property, where participants chose a single answer. An option "the experiment does not allow me to answer" was always available. Below are the detailed questions and possible answers, excluding the neutral option:

- The geodesics of this geometry have a shape like: For this question, participants had to pick one of the drawings of Figure 8 when answering about $\mathbb{H}^3$, and of Figure 9 when answering about $\mathbb{S}^3$.
- In this geometry, an object is generally visible: in a single location / it depends on the objects and their geometric characteristics / in two places, in front and behind you / in three places, forming a triangle around you / in five places, forming a pentagon around you.
- In this geometry, how can you estimate the distance from an object to you? by relying on its size when its distance from you varies. / item it is not possible to estimate it. / depending on its position relative to other objects in the environment.

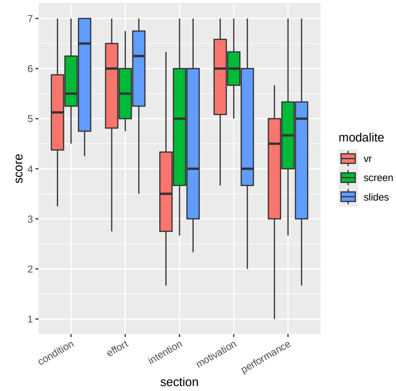

Figure 10: Box plot of UTAUT2 section results by section and modality.

- In this geometry, there are: no pair of special positions. / one pair of special positions. / all diametrically opposed position pairs play a special role. / a variable finite number of special position pairs. / a pair of positions visible in no direction.
- In this geometry, the regular polygons with right angles are: the triangle. / the square. / the pentagon. / polygons with more than five vertices. / there are no regular right-angled polygons.
- In this geometry, we can observe holonomy. What does this effect correspond to? a luminous halo around the objects. / a reflection of the environment. / a rotation of the environment. / distortion of distant objects.

5.4 Hypotheses and Results

Building on our literature review and the experimental protocol, we formulated the following hypotheses:
[H1] Globally, the VR modality will enable better progress on the mathematics questionnaire.

[H2] Participants will prefer the *vr* condition, as immersion in the virtual environment is more enjoyable than studying slides. The use of a headset enhances this sense of immersion compared to the on-screen version. This preference will be reflected in the hedonic motivation scores of the UTAUT questionnaire.

Data were analysed using non parametric statistics due to the limited number of participants.

Subjective Measures. The VRSQ questionnaire showed similar results across conditions and geometries. A Wilcoxon test revealed no statistically significant differences. The ratings (0-100) were low (*screen* $\overline{X} = 4.2$, $SD = 5.7$; *vr* $\overline{X} = 9.6$, $SD = 9.2$), indicating that cybersickness was not an issue.

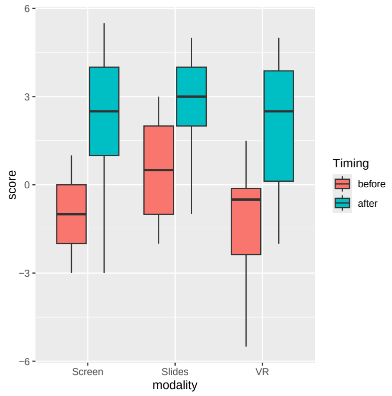

Figure 11: Box plot of mean scores in the math questionnaire before and after the experiment for each modality.

The subset of the UTAUT2 questionnaire gave similar results among conditions (*vr* $\overline{X} = 4.8$, $SD = 0.9$; *screen* $\overline{X} = 5.4$, $SD = 0.8$; *slides* $\overline{X} = 5.0$, $SD = 1.0$). We found no statistically significant difference while running a Kruskal-Wallis test (see Figure 10, $\chi^2 = 2.1$, df $= 2$, $p = 0.3$). We can still observe that motivation seems higher for the *vr* and *screen* conditions than for the *slides* condition (see Figure 10).

Objective Measures. To analyse the mathematics questionnaire, we assigned a score to each answer as follows: a correct answer is worth $+1$, a wrong answer is worth -1, and a neutral answer ("the experiment does not allow me to answer") is worth 0. The total score ranged from -6 to $+6$. Paired Wilcoxon tests revealed that each modality showed significantly better scores after the experiment than before ($p < 0.01$, see Figure 11).

To explore this further, we computed the mean score change before and after the experiment for each condition. For all participants, the *slides* condition increased the average score by 2.1 points, the *screen* condition, by 3 points, and the VR condition, by 3.3 points. We then computed this change separately for experts and novices. Experts showed slightly more progress, with their scores increasing by 2.4 points in the *slides* condition, 3 points in the *screen* condition, and 3.6 points in the VR condition, compared to novices, who saw increases of 1.9 points in the *slides* condition, 3 points in the *screen* condition, and 2.9 points in the *vr* condition. However a Kruskal-Wallis test did not reveal a statistically significant difference ($p = 0.4$) between the modalities.

The amount of time spent and the amount of displacement achieved in each scene (i.e. for each property) were analyzed using a logistic regression to determine whether they impacted the mathematics questionnaire scores. However, this analysis did not result in a model that could establish a correlation between time/displacement and score. This inability to construct such a model occurred because a large proportion of both correct and incorrect responses fell within the same time or displacement intervals, making it difficult to create a reliable model.

5.5 Discussion

The results show that all modalities led to progress, but the difference between the VR modality and the others was not statistically significant. Thus, we cannot confirm or reject our hypothesis **[H1]**.

Similarly, no significant difference was found in responses to the "hedonic motivation" section of the UTAUT questionnaire, although responses for VR and *screen* were higher and more clustered. Therefore, hypothesis **[H2]** cannot be confirmed or rejected.

To maximize comparability between the three modalities, we simplified the VR and screen experiments. We avoided complex interactions, which would have reduced comparability. However, this simplification likely missed unique features of each modality: physical interaction in VR, precise keyboard/mouse control on screen, and the limited interactivity of slides. One hypothesis for the small differences between modalities is this oversimplification.

Finally, based on math questionnaire scores and informal feedback, we found that the experiment required a higher level of mathematics than expected. The introductory document, meant to explain the vocabulary, used advanced formalism and concepts that prevented some participants from developing the expected skills.

6 CONCLUSION AND FUTURE WORKS

In this experiment we explored the differences between the three conditions *vr*, *screen* and *slides* for learning the properties of the non-Euclidean geometries $\mathbb{S}^3$ and $\mathbb{H}^3$. We have implemented a simulation of these geometries that allows them to be visualised and immersed inside both in virtual reality and on screen. We have also built a scenario that allows the key properties of these geometries to be tackled in a short space of time.

The results obtained encourage us to continue this work, in particular by reducing the presence of mathematical formalism, increasing the user's ability to interact with the system, and reinforcing the pedagogical approach. It is interesting to gain a better understanding of what virtual reality can and cannot do for mathematics education, particularly in areas as complex as non-Euclidean geometries. Virtual reality is a powerful tool that could enable any scientist in training to become more proficient at modelling and geometrising problems.

ACKNOWLEDGEMENTS

This work was supported in part by grants from CNRS 80 Prime ThurstonVR, DemoES AIR ANR-21-DMES-0001, Equipex+ Continuum ANR-21-ESRE-0030

REFERENCES

Baxter, G. and Hainey, T. (2019). Student perceptions of virtual reality use in higher education. *Journal of Applied Research in Higher Education*, 12.

Berger, P. (2015). Espaces Imaginaires. http://espaces-imaginaires.fr.

Buda, J. K. (2017). Integrating non-euclidean geometry into high school.

Caerols, H., Carrasco, R., and Asenjo, F. (2021). Using smartphone photographs of the moon to acquaint students with non-euclidean geometry. *American Journal of Physics*, 89.

Cevikbas, M., Bulut, N., and Kaiser, G. (2023). Exploring the benefits and drawbacks of ar and vr technologies for learners of mathematics: Recent developments. *Systems*, 11.

Coulon, R., Matsumoto, E., Segerman, H., and Trettel, S. (2022). Ray-marching thurston geometries. *Experimental Mathematics*, 31.

Eres, L., Besson, G., Boileau, M., Maillot, S., and Porti, J. (2010). Geometrisation of 3-manifolds. 13.

Gambini, A. (2021). Five years of comparison between euclidian plane geometry and spherical geometry in primary schools: An experimental study. *European Journal of Science and Mathematics Education*, 9.

Gambini, A. and Lénárt, I. (2021). Basic geometric concepts in the thinking of in-service and pre-service mathematics teachers. *Education Sciences*, 11(7).

Guerrero Idrovo, G., Ayala, A., Mateu, J., Casades, L., and Alaman, X. (2016). Integrating virtual worlds with tangible user interfaces for teaching mathematics: A pilot study. *Sensors*, 16.

Hedburg, J. and Alexander, S. (1994). Virtual reality in education: Defining researchable issues. *Educational Media International*, 31.

Kang, K., Kushnarev, S., Pin, W., Ortiz, O., and Shihang, J. (2020). Impact of virtual reality on the visualization of partial derivatives in a multivariable calculus class. *IEEE Access*, PP.

Kaufmann, H. and Schmalstieg, D. (2003). Schmalstieg, d.: Mathematics and geometry education with collaborative augmented reality. computers & graphics 27(3), 339-345. *Computers & Graphics*, 27.

Kim, H. K., Park, J., Choi, Y., and Choe, M. (2018). Virtual reality sickness questionnaire (vrsq): Motion sickness measurement index in a virtual reality environment. *Applied Ergonomics*, 69.

Kopczyński, E., Celińska, D., and Čtrnáct, M. (2017). HyperRogue: Playing with hyperbolic geometry. In *Proceedings of Bridges 2017: Mathematics, Art, Music, Architecture, Education, Culture*, Phoenix, Arizona. Tessellations Publishing.

Kotarinou, P. and Stathopoulou, C. (2017). *ICT and Liminal Performative Space for Hyperbolic Geometry's Teaching*.

Lai, J. W. and Cheong, K. H. (2022). Adoption of virtual and augmented reality for mathematics education: A scoping review. *IEEE Access*, 10.

Mikropoulos, T. and Natsis, A. (2011). Educational virtual environments: A ten-year review of empirical research (1999–2009). *Computers & Education*, 56.

Osypova, N., Kokhanovska, O., Yuzbasheva, G., and Kravtsov, H. (2021). *Implementation of Immersive Technologies in Professional Training of Teachers*, pages 68–90.

Su, Y.-S., Cheng, H.-W., and Lai, C.-F. (2022). Study of virtual reality immersive technology enhanced mathematics geometry learning. *Frontiers in Psychology*, 13.

Sukestiyarno, Y. L., Nugroho, K. U. Z., Sugiman, S., and Waluya, B. (2023). Learning trajectory of non-euclidean geometry through ethnomathematics learning approaches to improve spatial ability. *Eurasia Journal of Mathematics, Science and Technology Education*.

Takac, M. (2020). Application of web-based immersive virtual reality in mathematics education. In *2020 21th International Carpathian Control Conference (ICCC)*.

Thurston, W. P. (1986). Hyperbolic structures on 3-manifolds, i: Deformation of acylindrical manifolds. *Annals of Mathematics*, 124.

Velho, L., da Silva, V., and Novello, T. (2020). Immersive visualization of the classical non-euclidean spaces using real-time ray tracing in VR. In *Proceedings of Graphics Interface 2020*, GI 2020, pages 423–430.

Venkatesh, V., Thong, J., and Xu, X. (2012). Consumer acceptance and use of information technology: Extending the unified theory of acceptance and use of technology. *MIS Quarterly*, 36:157–178.

Wang, S.-T., Liu, L.-M., and Wang, S.-M. (2018). The design and evaluate of virtual reality immersive learning - the case of serious game "calcium looping for carbon capture". In *2018 International Conference on System Science and Engineering (ICSSE)*.

Weeks, J. Curved Spaces. a flight simulator for multiconnected universes, available from http://www.geometrygames.org/CurvedSpaces/.