Incentive Design in Hedonic Games with Permission Structures

Yuta Akahoshi, Yao Zhang, Kei Kimura, Taiki Todo and Makoto Yokoo
Graduate School of ISEE, Kyushu University, Fukuoka, Japan
{*akahoshi, zhang*}@*agent.inf.kyushu-u.ac.jp*, {*kkimura, todo, yokoo*}@*inf.kyushu-u.ac.jp*

Keywords: Hedonic Games, Diffusion Mechanism Design, Social Networks, Approximation Ratios.

Abstract: This paper investigates which coalition structure generation algorithms guarantee the incentive of agents to invite as many colleagues as possible in symmetric additively-separable hedonic games. We first clarify that, the incentive of invitation is not compatible with each of Nash stability and Pareto efficiency. Furthermore, we show that the worst-case ratio of social surplus achieved by any algorithm satisfying the incentive of invitation, compared to the best possible social surplus, is unboundedly small. We then introduce two problem restrictions to achieve somewhat positive results. More specifically, we showed that, when the utility graph of a hedonic game only contains three values, $\{-p, 0, p\}$, for some positive number p, there exists a polynomial-time algorithm to achieve both the incentive of invitation and 1/n-approximation with respect to the social surplus.

1 INTRODUCTION

A coalition formation game is one of the central problem in the field of multi-agent systems. Given set of multiple agents participating into the game, each of which has its own characteristics, it is desired to develop a *coalition structure generation algorithm* that partitions the whole set of agents into a certain number of subgroups (coalitions) that appropriately performs according to the given characteristics. Applications of such coalition formation problems includes, but not limited to, human resource allocations to jobs in the labor market and supply-chain management.

In the literature of coalition formation games, a *hedonic game* is well-studied, where each agent has a preference only over the set of coalitions in which he or she belongs to. An outcome, returned by a coalition structure generation algorithm, is usually evaluated based on various criteria related to some kind of *stability*, including Nash stability, individual stability, and individual rationality. Clarifying the complexity of showing the existence of stable outcomes has been one of the research trends in the domain of computational social choice in last few decades.

In practical situations, however, the set of participating agents is not given a priori. Instead, agents usually invite their colleagues to participate in the decision making. If we assume that each agent prefers having more members in his/her coalition, such an invitation process will be naturally incentivized. On the other hand, if agents dislike some others, agents may pretend inviting colleagues to the decision making. The main purpose of this paper is to design *incentive-compatible* coalition structure generation algorithms, which incentivizes agents to invite as many colleagues as possible to the decision making. Such a research direction has recently been called as *permission structures* in the field of coalitional games and operations research, and as *diffusion mechanism design* in the field of multi-agent systems and artificial intelligence.

As a first application of the perspective of diffusion mechanism design to the literature of hedonic games, we focus on a special class of hedonic games, so called as *symmetric additively-separable hedonic games (SASHG)* (Burani and Zwicker, 2003). In an SASHG, there is an undirected weighted graph, in which the set of vertices corresponds to the set of agents. Given a coalition, each agent's *utility* is given as the sum of the weights of the edges connecting the agent to other members in the coalition. Each agent then prefers coalitions that give her higher utilities.

At the same time, the set of agents has a *permission structure*, given as a directed unweighted graph, in which the set of vertices again corresponds to the set of agents. Intuitively, the existence of a directed edge from agent i to agent j in a permission structure means that agent i has a right to invite agent j to the SASHG. We then call agent i as a parent of agent j, and agent j as a child of agent i. Technically speak-

Akahoshi, Y., Zhang, Y., Kimura, K., Todo, T. and Yokoo, M.
Incentive Design in Hedonic Games with Permission Structures.
DOI: 10.5220/0013320400003890

In *Proceedings of the 17th International Conference on Agents and Artificial Intelligence (ICAART 2025) - Volume 1*, pages 184-195
ISBN: 978-989-758-737-5; ISSN: 2184-433X

ing, we consider two variants of incentives of invitations, namely the *conjunctive* variant and the *disjunctive* variant. In the conjunctive variant, an agent participates if all her parent invite her. On the other hand, in the disjunctive variant, an agent participates if at least one of her parents invites her.

To sum up, the problem we tackle in this paper is the symmetric additively-separable hedonic games with permission structures, SASHG-PS in short. An SASHG-PS instance has two graphs, one of which is defines an SASHG, and the other defines its permission structures. Given an SASHG-PS instance, a coalition structure generation algorithm returns a coalition structure, hopefully guaranteeing incentives for agents to invite as many colleagues as possible, along with some stability properties. The main focus in this paper is then to clarify the existence of such coalition structure generation algorithms for SASHG-PS.

In this paper, we first give a formal definition of the new class of games, SASHG-PS. We then show three impossibility results and one possibility result, according to various combinations of stability, fairness, and efficiency properties. We also investigate approximating the social surplus, defined as the sum of the utilities of participating agents, by incentive-compatible algorithms. On this direction, we first provide a general impossibility result, and then introduce two problem restrictions under which some positive results have been clarified.

2 RELATED WORKS

Hedonic games are a special class of coalition formation games, in which each agent only cares about which members does she have in the same coalition with her and ignores the structures of other coalitions (Dreze and Greenberg, 1980; Bogomolnaia and Jackson, 2002). Gairing and Savani (2019) focused on *symmetric additively-separable* hedonic games and provided several complexity results on finding various stable outcomes. Flammini et al. (2021) considered additively-separable hedonic games from the viewpoint of strategy-proof mechanism design. The main difference with this paper is that, while this paper focuses on information diffusion strategies by agents, they considered the cases where agents can misreport their valuation functions. A survey on hedonic games by Aziz and Savani (2016) summarizes solution concepts and known results on several variants of hedonic games, from the viewpoints of algorithms and computational complexity.

The research of diffusion mechanism design, also known as mechanism design over social networks, was initiated by Li et al. (2017), which considered single-item auctions and proposed a strategy-proof mechanism. After that, several works investigated strategy-proof resource allocation mechanisms with monetary compensations, e.g., multi-unit auctions and redistributions (Zhao et al., 2018; Kawasaki et al., 2020; Li et al., 2020; Zhang et al., 2020; Jeong and Lee, 2024). On the other hand, there is limited research on decision making without money from the perspective of diffusion mechanism design. Recently, Kawasaki et al. (2021) and You et al. (2022) considered diffusion mechanism design for house allocation problems, which do not allow monetary compensations. Cho et al. (2022) studied two-sided matching over social networks, and Ando et al. (2024) studied facility location games. However, as far as the authors know, this paper is a very first attempt to study hedonic games from the perspective of diffusion mechanism design.

While the paradigm of diffusion mechanism design has recently been introduced in the field of computer science, a quite similar approach has also been considered in the field of operations research, which is known as *cooperative games with permission structures*. The two variants of cooperative games with permission structures, namely the conjunctive variant (Gilles et al., 1992) and the disjunctive variant (van den Brink, 1997), has been originally introduced for the domain of general coalitional games. The main difference between the diffusion mechanism design and the cooperative games with permission structures is that, while the former considers the decision making process from the viewpoint of mechanism design, the latter is defined consistently with the traditional cooperative games. The main purposes of these two directions are therefore a bit different; diffusion mechanism design focuses on designing algorithms/mechanisms that incentivises agents to invite other agents, and the cooperative games with permission structures focuses on analysing the existence of appropriate solution concepts and their characterization/axiomatizations. For more detail on permission structures, please refer to a recent survey on games with permission structures by van den Brink (2017).

3 MODEL

As explained in Section 1, the problem we consider in this paper is defined as a combination of two well-studied models, namely the symmetric additively-separable hedonic games and the coalition games with permission structures. Now we define the model of symmetric additively-separable hedonic games with permission structures (SASHG-PS).

Let N be the set of all the potential agents, and let $n := |N|$ be the number of potential agents. A subset $C \subseteq N$ is called as a *coalition*. Given agent $i \in N$, let N_i be the set of coalitions that contains i, i.e.,

$$N_i := \{C \subseteq N \mid C \ni i\}$$

Also, given N, a partition $\pi = \{C_1, C_2, \ldots\}$ of N is called as a *coalition structure*, such that

$$\forall k, \ell (\neq k), C_k \cap C_\ell = \emptyset$$

and

$$\bigcup_{k=1,2,\ldots} C_k = N.$$

Let Π_N be the set of all the possible coalition structures that partitions the set N of agents. Furthermore, given coalition structure $\pi \in \Pi_N$ and agent $i \in N$, let $\pi(i)$ be the coalition in π in which agent i belongs, i.e.,

$$\pi(i) := C \in \pi \text{ s.t. } C \ni i.$$

There is a *utility graph* $G(N,E,w)$, which is an undirected edge-weighted graph, where $w_e \in \mathbb{R}$ denotes the weight of edge $e \in E$. For notation simplicity, we assume $w_e \neq 0$, i.e., the edges with a zero weight is removed from the set E of edges. When an edge $e \in E$ has agent (vertex) $i \in N$ in one of its endpoints, we denote $e \ni i$. Furthermore, when both of the endpoints of edge e belongs a coalition C, we denote $e \in C$. Given utility graph $G(N,E,w)$ and agent $i \in N$, a symmetric additively-separable preference $\succsim_i$ of agent i, over the set N_i of coalitions, is defined as follows:

$$\forall C, D \in N_i, C \succsim_i D \Leftrightarrow \sum_{e \ni i | e \in C} w_e \geq \sum_{e \ni i | e \in D} w_e.$$

Also, given utility graph $G(N,E,w)$, agent $i \in N$, and a coalition structure $\pi \in \Pi_N$, agent i's utility is defined as

$$u_i(\pi) := \sum_{e \ni i | e \in \pi(i)} w_e.$$

SASHG-PS also has another graph, the *permission graph* $H(N,A)$, represented as an unweighted directed acyclic graph. The set of vertices corresponds to the set of agents N. Also, the set $A \subseteq N \times N$ of directed edges is assumed not to contain multi-edges and self-loops. The existence of directed edge $(i, j) \in A$ means that agent i can send an invitation to agent j. Given permission graph $H(N,A)$, the set $O \subseteq N$ of vertices that have no incoming edges is referred to as the *original agents*, who is assumed to participate in the game even without any invitation. Given permission graph $H(N,A)$ and agent $i \in N$, let $\delta_i \in \mathbb{N}_{\geq 0}$ be the *distance* of agent i, defined as the minimum distance from an original agent, i.e.,

$$\delta_i := \min_{o \in O}$$

Also, given permission graph $H(N,A)$ and agent $i \in N$, let $S(i)$ be the set of agents (vertices) that agent i can send an invitation, i.e.,

$$S(i) := \{j \in N \mid (i, j) \in A\}$$

An important note here is that, although both the utility graph and the permission graph are components of an SASHG-PS instance, these are not directly observed by a coalition structure generation algorithm. Instead, agents first decide who to invite to the game, and then the algorithm is able to observe who invite which agents and finally who participate in the game.

To sum up, an instance $I(G,H)$ of SASHG-PS has two graphs, the utility graph $G(N,E,w)$ and the permission graph $H(N,A)$, where the set N of potential agents is in common in these two graphs. Now let us explain how an SASHG-PS proceeds.

1. The Nature determines both the utility graph and the permission graph, while the both are not directly observable by anyone.
2. A coalition structure generation algorithm is publicly announced.
3. Observing the description, i.e., the rule of a game, each agent decide which colleagues to invite to the game.
4. According to the invitation strategies of the agents, the set of participating agents is fixed. Then, the algorithm returns the outcome.

3.1 Optimization Criteria

Before introducing several properties related to stability, fairness, and efficiency, we first define an evaluation criteria for quantitatively measuring the worst-case performance of coalition structure generation algorithms. A *social surplus*, defined as the sum of the utilities of all the participating agents, is a well-known evaluation criteria in the field of micro-economics.

Definition 1 (Social Surplus)**.** *Given SASHG-PS instance, a* social surplus $\mathrm{SW}(\pi)$ *of a coalition structure* $\pi \in \Pi_N$ *is defined as*

$$\mathrm{SW}(\pi) := \sum_{i \in N} u_i(\pi_i).$$

We then define the approximation ratio for the social surplus, which enables evaluating the worst-case performance of (potentially incentive-compatible) algorithms, compared to the global optimal solution that is not necessarily incentive-compatible.

Table 1: Comparisons of four incentive-compatibility properties. Properties on top-left direction is more demanding. No inclusion relation exists between D-DSIC and C-EPIC.

		Equilibrium Concept	
		Dominant Strategy	Ex-Post
Permission Type	Conjunctive	C-DSIC	C-EPIC
	Disjunctive	D-DSIC	D-EPIC

Definition 2 (Approximation Ratio)**.** *Given SASHG-PS instance I, an optimal coalition structure* $\pi^*(I) \in \Pi_N$ *is a coalition structure that has the maximum social surplus, i.e.,*

$$\pi^*(I) := \arg\max_{\pi \in \Pi_N} \mathrm{SW}(\pi).$$

An approximation ratio $\alpha_{\mathcal{M}}$ *of a coalition structure generation algorithm* $\mathcal{M}$ *is the minimum ratio of the social surplus of a coalition structure obtained by the algorithm, against the optimal social surplus. Formally,*

$$\alpha_{\mathcal{M}} := \min_{I} \frac{\mathrm{SW}(\mathcal{M}(I))}{\mathrm{SW}(\pi^*(I))}.$$

The optimal approximation ratio is one, and any algorithm has an approximation ratio weakly less than one. Designing coalition structure generation algorithms having a larger approximation ratio is desirable. If there is an algorithm achieving the approximation ratio of one, we call it optimal.

3.2 Properties

In this subsection we define several properties that coalition structure generation algorithms should satisfy. We first define incentive compatibilities. According to the characteristics of SASHG-PS, specifically the permissions' types and the equilibrium concepts, we define four variants of incentive compatibility properties, summarized in Table 1.

Definition 3 (Incentive Compatibility)**.** *Given SASHG-PS instance* $I(G,H)$*, a coalition structure generation algorithm* $\mathcal{M}$ *is said to satisfy*

- dominant-strategy incentive compatibility for conjunctive permissions *(C-DSIC) if, for each agent* $i \in N$*, inviting as many colleagues in* $S(i)$ *as possible is a* dominant strategy*, under the assumption that an agent participates in the game if and only if* all her parents *invite her.*
- dominant-strategy incentive compatibility for disjunctive permissions *(D-DSIC) if, for each agent* $i \in N$*, inviting as many colleagues in* $S(i)$ *as possible is a* dominant strategy*, under the assumption that an agent participates in the game if and only if* at least one of her parents *invites her.*
- ex-post incentive compatibility for conjunctive permissions *(C-EPIC) if, for each agent* $i \in N$*, inviting as many colleagues in* $S(i)$ *as possible is a* best strategy when all the other participating agents invite as many colleagues as possible*, under the assumption that an agent participates in the game if and only if* all her parents *invite her.*
- ex-post incentive compatibility for disjunctive permissions *(D-EPIC) if, for each agent* $i \in N$*, inviting as many colleagues in* $S(i)$ *as possible is a* best strategy when all the other participating agents invite as many colleagues as possible*, under the assumption that an agent participates in the game if and only if* at least one of her parents *invites her.*

By definition, C-DSIC implies both D-DSIC and C-EPIC, and both D-DSIC and C-EPIC imply D-EPIC. Between D-DSIC and C-EPIC, there is no inclusion relation in general.

We next define stability, fairness, and efficiency properties that are quite popular in the literature of hedonic games. Both individual rationality and Nash stability have been known as a stability property, where the latter implies the former.

Definition 4 (Individual Rationality)**.** *Given SASHG-PS instance* $I(G,H)$*, a coalition structure* $\pi \in \Pi_N$ *is said to be* individually rational *if*

$$\forall i \in N, u_i(\pi(i)) \geq 0$$

holds. A coalition structure generation algorithm $\mathcal{M}$ *is said to satisfy individual rationality (IR) if* $\mathcal{M}(I)$ *is individually rational for any SASHG-PS instance I.*

Definition 5 (Nash Stability)**.** *Given SASHG-PS instance* $I(G,H)$*, a coalition structure* $\pi \in \Pi_N$ *is said to be* Nash stable *if*

$$\forall i \in N, u_i(\pi(i)) \geq u_i(C \cup \{i\})$$

holds for any $C \in \pi \cup \{\emptyset\}$*. A coalition structure generation algorithm* $\mathcal{M}$ *is said to satisfy Nash stability (NS) if* $\mathcal{M}(I)$ *is Nash stable for any SASHG-PS instance I.*

Pareto efficiency is one of the most popular efficiency property in the field of economics. Intuitively, it requires that there is no other outcome that is better for all the participating agents.

Definition 6 (Pareto Efficiency)**.** *Given SASHG-PS instance $I(G,H)$, a coalition structure $\pi \in \Pi_N$ is said to be* Pareto efficient *if there is no other coalition structure $\pi' \in \Pi_N$ such that both*

$$\forall i \in N, u_i(\pi'(i)) \geq u_i(\pi(i))$$

and

$$\exists j \in N, u_j(\pi'(j)) > u_j(\pi(j))$$

hold. A coalition structure generation algorithm $\mathcal{M}$ is said to satisfy Pareto efficiency (PE) if $\mathcal{M}(I)$ is Pareto efficient for any SASHG-PS instance I.

Envy-freeness is a fairness property, which requires that for each agent, changing her position with any other agent is not beneficial.

Definition 7 (Envy-Freeness)**.** *Given SASHG-PS instance $I(G,H)$, a coalition structure $\pi \in \Pi_N$ is said to be* envy-free *if*

$$\forall i, j \in N \text{ s.t. } \pi(i) \neq \pi(j),$$
$$u_i(\pi(i)) \geq u_i((\pi(i) \setminus \{j\}) \cup \{i\})$$

holds. A coalition structure generation algorithm $\mathcal{M}$ is said to satisfy envy-freeness *(EF) if $\mathcal{M}(I)$ is envy-free any SASHG-PS instance I.*

In the literature of hedonics games, there are several other properties, including individual stability, contractually individual stability, the core, and the strict core. For more detail on these solution concepts, please refer to a survey chapter by Aziz and Savani (2016).

4 (IM)POSSIBILITIES

We are now ready to present our technical contribution. We first show that, although all the properties introduced in the previous section seems to be desirable, various combinations among them result in the non-existence of algorithms. On the other hand, Theorem 4 shows that just keeping every participating agent as a singleton satisfies IR, EF, and C-DSIC.

Theorem 1. *There exists no coalition structure generation algorithm that simultaneously satisfies D-EPIC and NS.*

Proof. Consider an SASHG-PS instance described in Fig. 1, and focus on the strategic invitation by agent a. When agent a invite agent c, all the three agents

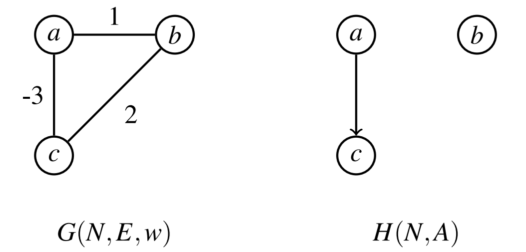


Figure 1: An SASHG-PS instance for which D-EPIC and NS are incompatible, used in the proof of Theorem 1.

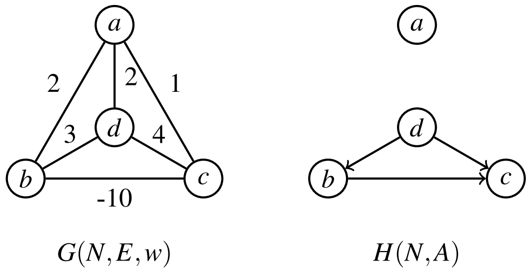


Figure 2: An SASHG-PS instance for which PE and either D-DSIC or C-EPIC are incompatible, used in the proof of Theorem 2.

participate. Then, the unique Nash stable coalition structure is

$$\pi = \{\{a\}, \{b,c\}\},$$

under which agent a's utility is 0.

When agent a does not invite agent c, there are only two participating agents a and b. Then, the unique Nash stable coalition structure is

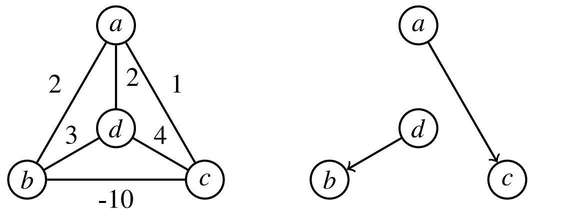
$$\pi = \{\{a,b\}\},$$

under which agent a's utility is 1. Thus, agent a is better off by not inviting agent c. Since the permission graph $H(N,A)$ has only one directed edge, the weakest variant of IC condition, namely D-EPIC, is violated. □

Theorem 2. *There exists no coalition structure generation algorithm that simultaneously satisfies PE and either D-DSIC or C-EPIC.*

Proof. Consider an SASHG-PS instance described in Fig. 2, in which there are five Pareto efficient coalition structures:

1. $\{\{a,b\},\{c,d\}\}$
2. $\{\{a,c\},\{b,d\}\}$
3. $\{\{a,b,d\},\{c\}\}$
4. $\{\{a,c,d\},\{b\}\}$
5. $\{\{a,b,c,d\}\}$



$G(N,E,w)$ $H(N,A)$

Figure 3: An SASHG-PS instance for which PE, IR, and D-EPIC are incompatible, used in the proof of Theorem 3.

When a Pareto efficient algorithm returns coalition structure 1., 2., 3., or 5. above, agent b's utility is strictly less than 5. Therefore, agent b has an incentive not to invite agent c; the unique Pareto efficient coalition structure is $\{\{a,b,d\}\}$ if agent c does not participate, in which agent b's utility is 5.

When a Pareto efficient algorithm returns coalition 4. above, agent d's utility is 5. Therefore, agent d has an incentive not to invite agent b; the unique Pareto efficient coalition structure is $\{\{a,c,d\}\}$ if agent b does not participate, in which agent d's utility is 6.

The existence of these two beneficial manipulations violates both D-DSIC and C-EPIC (and thus C-DSIC), which completes the proof. Note that D-EPIC is not violated in this instance, since agent c is not excluded by any sole manipulation by either agent b or d. □

Theorem 3. *There exists no coalition structure generation algorithm that simultaneously satisfies PE, IR, and D-EPIC.*

Proof. Consider an SASHG-PS instance described in Fig. 3, in which there are four coalition structures that is both Pareto efficient and individually rational:

1. $\{\{a,b\},\{c,d\}\}$
2. $\{\{a,c\},\{b,d\}\}$
3. $\{\{a,b,d\},\{c\}\}$
4. $\{\{a,c,d\},\{b\}\}$

When a Pareto efficient algorithm returns coalition structure 1., 2., or 4. above, agent a's utility is strictly less than 4. Therefore, agent a has an incentive not to invite agent c; the unique Pareto efficient coalition structure is $\{\{a,b,d\}\}$ if agent c does not participate, in which agent a's utility is 4.

When a Pareto efficient algorithm returns coalition structure 3. above, agent d's utility is 5. Therefore, agent d has an incentive not to invite agent b; the unique Pareto efficient coalition structure is $\{\{a,c,d\}\}$ if agent b does not participate, in which agent d's utility is 6.

The existence of these two beneficial manipulations, which are independent with each other, implies that D-EPIC is violated. Thus, IR, PE and D-EPIC are incompatible. □

Theorem 4. *There exists a coalition structure generation algorithm that simultaneously satisfies IR, EF, and C-DSIC.*

Proof. Consider a coalition structure generation algorithm $\mathcal{M}$ that returns, for any SASHG-PS instance, a coalition structure in which every agent constructs a singleton coalition.

In such an algorithm, every agent has utility zero for any instance, regardless of the invitation strategies of all the agents. Thus, $\mathcal{M}$ satisfies C-DSIC, the strongest incentive compatibility, and individual rationality

Furthermore, when a coalition structure $\pi \in \Pi_N$ is such that every agent constructs a singleton coalition, it holds that

$$\forall i,j \in N, (\pi(j) \setminus \{j\}) \cup \{i\} = \pi(i).$$

Thus, the condition of envy-freeness is satisfied. □

5 DESIGNING APPROXIMATION ALGORITHMS

The impossibility results presented in the previous section, regarding the incentive compatibility properties in SASHG-PS, are quite negative. Now we focus only on the incentive compatibility and analyze the worst-case performance of incentive compatible algorithms with respect to the social surplus.

The theorem below has a quite negative implication. The property D-EPIC is the weakest among the four definitions. This theorem states that, even with the weakest incentive requirement on strategic invitation, there is a chance that the achievable social surplus becomes nearly zero.

Theorem 5. *The approximation ratio of any coalition structure generation algorithm satisfying D-EPIC is zero.*

Proof. Consider an SASHG-PS instance described in Fig. 4, where the weight m of edge between agents b and c is sufficiently large. The optimal coalition structure for this instance is the grand coalition, i.e., all the three agents belong to the same coalition, in which the social surplus is $m+1$.

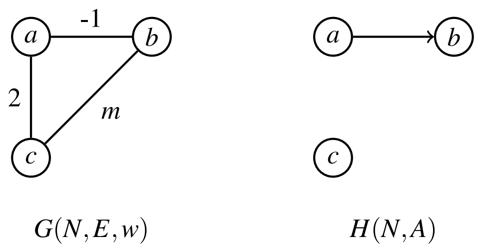


Figure 4: An SASHG-PS instance for which D-EPIC and NS are incompatible, used in the proof of Theorem 5.

Since m is arbitrarily large, any coalition structure generation algorithm that has bounded approximation ratio must put both agents b and c in a single coalition. Under such a coalition structure, agent a's utility is at most 1.

Also, when agent a does not invite agent b, the optimal coalition structure for two agents a and c is the grand coalition that contains both agents a and c in a single coalition, in which the social surplus is 2. Thus, any coalition structure generation algorithm that has bounded approximation ratio must put both of these two agents a and c in a single coalition, when agent b does not participate.

Therefore, agent a has an incentive not to invite agent b; agent a's utility is 2 when she does not invite agent b, while it is at most 1 when she invite agent b. Since agent a can solely decide the participation of agent b, such an algorithm violates D-EPIC, the weakest incentive compatibility. □

6 ALGORITHMS FOR RESTRICTED GRAPHS

In the previous section we have presented a negative results on the approximability of social surplus by incentive compatible coalition structure generation algorithms. A natural question is then how can we overcome the negative results by introducing some problem restriction to SASHG-PS.

A quite trivial case is such that the permission graph $H(N,A)$ has no edge, i.e., $A = \emptyset$, where any incentive compatibility constraint becomes void and we can simply apply known algorithms in the literature. However, when the permission structure is not observable a priori, which seems to be very likely in practical situations, assuming such a restriction is not quite reasonable.

We therefore focus on the restriction to the utility graph in this paper. At first, we provide a sufficient condition to guarantee an optimality of the social surplus. Namely, when the edges with non-zero weights of a utility graph construct a forest, there exists an optimal coalition structure generation algorithm that also satisfies C-DSIC.

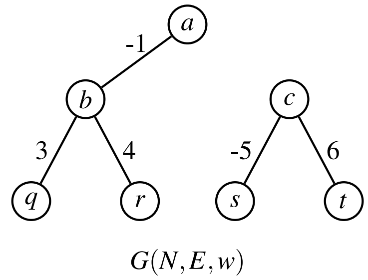


Figure 5: An example of forest utility graph.

Theorem 6. *There exists an optimal C-DSIC coalition structure generation algorithm when the utility graph is given as a forest.*

Proof. Consider a coalition structure generation algorithm $\mathcal{M}$ that (i) first remove all the negative edges from the utility graph G and then (ii) choose each of the connected component in the modified utility graph as a coalition. Note that, when the original utility graph G is assumed to be a forest, it is still a forest after the removal of all the negative edges. Obviously, it returns the optimal coalition structure for such a forest graph.

Now let us show that this algorithm $\mathcal{M}$ satisfies C-DSIC. In this algorithm, agent i belongs to the same coalition with any agent j if and only if there is an edge with a strictly positive weight. Here, for any whole set N of agents, any agent $i \in N$, and any subset $N' \subseteq N$ such that $N' \ni i$, let $\pi \in \Pi_N$ and $\pi' \in \Pi_{N'}$ be the coalition structure provided by $\mathcal{M}$ for each of two participating sets, N and N', of agents, respectively. Then, we have

$$u_i(\pi(i)) \geq u_i(\pi'(i)),$$

which implies C-DSIC. □

Example 1. *Now we demonstrate how the algorithm $\mathcal{M}$ proposed in the proof of Theorem 6 works for the SASHG-PS instance described in Figure 5.*

The algorithm first removes all the edges with negative weights in the utility graph $G(N,E,w)$. In the figure, there are two such edges, between agents a and b, and between agent c and s. Then, the algorithm returns each of the connected components as a coalition. The outcome is therefore,

$$\{\{a\},\{b,q,r\},\{c,t\},\{s\},\}$$

which is clearly optimal as it contains all the edges with positive weights and does not contain any edge with a negative weight.

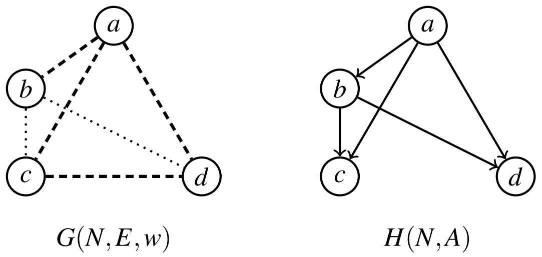


Figure 6: An SASHG-PS instance showing the upper bound of approximation ratio for the case of $n = 3$, used in the proof of Theorem 7. The dashed edge has a positive weight $+p$, while the dotted edge has a negative weight $-p$.

7 ALGORITHMS FOR RESTRICTED WEIGHTS

Restricting the domain of weights of the utility graph is also a natural approach to overcome impossibility results in the literature. Even for our problem of SASHG-PS, there is a quite trivial case where such a restriction works. For example, when the weights of the utility graph is assumed to be non-negative, in which returning the grand coalition is the optimal.

Here we provide a quite different condition to guarantee the tight approximability results for the social surplus, by either D-DSIC or C-EPIC coalition structure generation algorithms. Namely, while we allow the existence of negative weights, all the non-zero weights have the same absolute value. We first show, in Theorem 7, that any coalition structure generation algorithm satisfying either D-DSIC or C-EPIC has an approximation ratio of at most $\frac{1}{n}$ for $n = 3$, and at most $\frac{1}{n-1}$ for $n \geq 4$, which works as an upper bound. We then show, in Theorem 8, the existence of D-DSIC algorithm which has an approximation ratio of $\frac{1}{n}$ for odd n and $\frac{1}{n-1}$ for even n. That is, the approximation ratio obtained in this section is asymptotically tight. We further show that the proposed algorithm runs in polynomial-time.

Theorem 7. *Assume that the weights of the utility graph are restricted to $\{-p, 0, p\}$ for some constant $p \in \mathbb{R}_+$. Then, the approximation ratio of any coalition structure generation algorithm satisfying either D-DSIC or C-EPIC is at most $1/n$ for $n = 3$, and $\frac{1}{n-1}$ for $n \geq 4$.*

Proof. We first show the upper bound $\frac{1}{n}$ for the case of $n = 3$, and then show the upper bound $\frac{1}{n-1}$ for the case of $n \geq 4$.

To show the bound for $n = 3$, we begin with an SASHG-PS instance with $n = 4$, described in Figure 6. Note that the dashed edges in the figure have a positive weight $+p$, and the dotted edges have a negative weight $-p$.

Figure 7: An SASHG-PS instance obtained by removing both agents c and d from Figure 6.

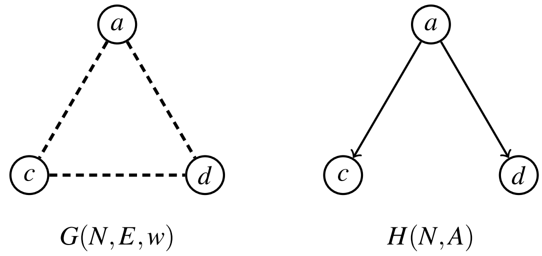


Figure 8: An SASHG-PS instance obtained by removing agents d from Figure 6.

Consider that agent b is now willing to strategically invite her colleagues, agents c and d. When all those four agents participate, agent b's utility would be p in coalition $\{a,b\}$, and 0 in any other coalition. If an algorithm returns a coalition structure where agent b does not belong to coalition $\{a,b\}$, agent b has an incentive not to invite both agents c and d, resulting in the graphs shown in Figure 7; agent b's utility would then be 1, to guarantee that the approximation ratio is bounded. The existence of such an incentive violates both C-EPIC (where agent b can solely prevent the participation of both c and d) and D-DSIC (where agent b can still prevent the participation of agents c and d *if* agent a also chooses not to invite them). Therefore, in the case of Figure 6, an algorithm with a bounded approximation ratio must return a coalition structure containing a coalition $\{a,b\}$ as its member. Note that in this coalition, agent a's utility is also 1.

Now let us consider that agent a is also willing to strategically invite agent b (see Figure 8). When agent b does not participate, agent a's utility cannot exceed 1; otherwise agent a has an incentive to prevent agent b'2 participation, violating D-EPIC. Therefore, the set of possible coalition structures in Figure 8 is one of the followings:

$$\{\{a,c\},\{d\}\}, \quad \{\{a,d\},\{c\}\},$$
$$\{\{a\},\{c,d\}\}, \quad \{\{a\},\{c\},\{d\}\}.$$

The maximum social surplus is then 1, while the optimal social surplus is 3. Thus the target ratio 1/3 is proved.

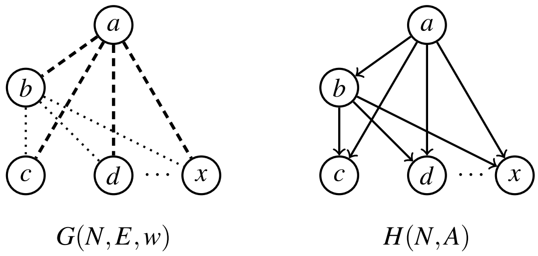


Figure 9: An SASHG-PS instance showing the upper bound of approximation ratio for the case of $n \geq 4$, used in the proof of Theorem 7.

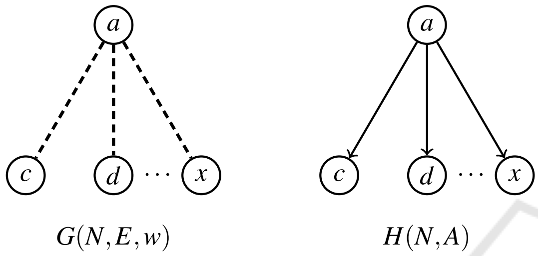


Figure 10: An SASHG-PS instance obtained by removing agent b from Figure 9.

Now we turn to show that the upper bound is $\frac{1}{n-1}$ for the case of $n \geq 4$. Consider the case described in Figure 9, where there are $n-1$ agents, agent a is connected to all the other agents, b, c, d, $\cdots$, x, with an edge with a positive weight $+p$ in the utility graph G, while agent b is connected to those agents c, d, $\cdots$, x, with an edge with a negative weight $-p$ (except for agent a). Also, in the permission graph H, both of agents a and b are able to invite those agents to which they are connected to in G, respectively.

Since agent b has only one edge with a positive weight, connected with agent a, agent b's utility is 1 if and only if coalition $\{a,b\}$ is in the coalition structure, and zero otherwise. From a similar argument with the case of $n=3$, coalition $\{a,b\}$ must be formed for the case of Figure 9; otherwise both C-EPIC and D-DSIC are violated. Note that in this case agent a's utility is 1.

We then consider that agent a is also willing to strategically invite agent b (see Figure 10, where there are n agents). Again, from a similar argument with the case of $n=3$, agent a's utility cannot exceed 1; otherwise D-EPIC is violated. Therefore, the possible coalition structures in Figure 10 are such that agent a forms a coalition with at most one another agent. The social surplus is then at most 1, while the optimal social surplus is $n-1$. Thus the target ratio $\frac{1}{n-1}$ is proved. □

Theorem 8. *Assume that the weights of the utility graph are restricted to $\{-p, 0, p\}$ for some constant $p \in \mathbb{R}_+$. Then, there is a polynomial-time coalition structure generation algorithm that satisfies D-DSIC and has an approximation ratio of $1/n$ when n is odd, and $\frac{1}{n-1}$ when n is even.*

Proof. The theorem is separately shown in the two propositions below, Propositions 1 and 2. □

We now propose a polynomial-time algorithm that satisfies D-DSIC and has an asymptotically optimal approximation ratio. Briefly speaking, the proposed algorithm tries to find a maximum matching by focusing on the edges with a positive weights. Obviously, such a maximum matching can be found in polynomial time. However, a naïve implementation of maximum matching algorithm fails to satisfy even D-EPIC, which will be explained later in this section.

Definition 8 (Proposed Algorithm)**.** *Given SASHG-PS instance I, runs the following procedure:*

1. *Initialization: Remove all the edges with a negative weight in the utility graph. Also, label all the edges with a positive weight in the utility graph with* pre-update*. Let n be the number of participating agents, and let $h = 1$.*
2. *Construct a linear order $\triangleright$ of all the participating agents so that, for any two agents i, j, if either*
   - $\delta_i < \delta_j$*, or*
   - $\delta_i = \delta_j$ *and* $|S(i)| > |S(j)|$

   holds, then we order $i \triangleright j$, i.e., agent i is more prioritized than agent j. When both $\delta_i = \delta_j$ and $|S(i)| = |S(j)|$ holds, we break the tie based on a consistent way under which agents' invitation strategies does not affect.
3. *For the most prioritized remaining agent i in the order $\triangleright$:*
   - *Update the weight of all edges $e \ni i$ that have labeled as* pre-update *(if any) with the weight 2^{n-h}, and then label them with* post-update.
   - *Let $h \leftarrow h+1$, remove i from the top of the order $\triangleright$, and go to 3. if $h \leq n$; otherwise go to 4.*
4. *Obtain a maximum weight matching for the utility graph with updated weights, where the tie-breaking among the edges with the same (updated) weight is determined based on the linear order $\triangleright$.*

Intuitively, the algorithm works like a "serial dictatorship mechanism" in the literature of strategy-proof mechanism design, which is a powerful tool to guarantee incentive compatibility in various decision making situations. First, the most prioritized agent, say i, in $\triangleright$ chooses the whole set of outcomes in which agent i is matched with another agent. Among them, the second prioritized agent, say j, chooses a subset in which agent j is also matched.

It is clear that the proposed algorithm runs in a polynomial-time. Step 1. completes in $O(E)$. Step 2. is just sorting the agents based on the information available from the algorithm, which runs in $O(N)$. Updates in Step 3. occur E times, since every edge is updated exactly once. Finally, in Step 4., we obtain the maximum weight matching in a polynomial-time, e.g., in $O(N^2E)$ by Edmonds' algorithm (Edmonds, 1965). Since these steps are processed in a series, the total runtime is $O(N^2E)$.

We now demonstrate how the proposed algorithm works, by applying it to the case of Figure 9.

Example 2. *Consider the situation described in Figure 9, and assume that there are three agents in the list d, ..., x. That is, there are totally six agents. In Step 1., the algorithm removes all the negative edges, and label all the positive edges with* pre-update. *Then, in Step 2., the participating agents will be linearly ordered. When all the agents participates, the order is, say,*

$$a \rhd b \rhd c \rhd d \rhd \cdots \rhd x.$$

After that, in Step 3., the weights are updated. Now that all the remaining positive edges has agent a at one of their endpoints, all the weights are updated as 2^5.

By applying a maximum weight matching for this modified utility graph, with the tie-breaking rule based on $\rhd$*, the edge between a and b is chosen, and the algorithm terminates. From agent a's perspective, removing agent b does not help, since agent a is already satisfied by being connected with an agent. Note that the utility is calculated based on the original weights of the utility graph.*

From agent b's perspective, any removal of her colleagues does not help, since agent b is also connected. Note that, if an arbitrary tie-breaking rule would be used for achieving a maximum weight matching in Step 4., it may possible that agent c is matched with agent a. In such a case, agent b has an incentive not to invite agent b (and possibly all the others, so that they cannot participate if agent a also does not invite them), violating D-DSIC.

We first briefly explain why the proposed algorithm satisfies D-DSIC. Due to space limitations, we present a proof sketch. Also, Example 2 above will also help the readers to understand the intuition why we need to carefully choose the tie-breaking methods.

Proposition 1. *When the weights are restricted to* $\{-p, 0, p\}$ *for some constant* $p \in \mathbb{R}_+$*, algorithm 8 satisfies D-DSIC.*

Proof Sketch. By definition, the linear order $\rhd$ satisfies the property such that, for any agent i, removing any of her colleagues from the games never changes the order of the agents who are originally more prioritized than agent i. Furthermore, as we have mentioned in Example 2, using the tie-breaking rule based on $\rhd$ is essential to guarantee D-DSIC. Therefore, there exists no case in which agent i can get matched and receives utility $+p$ by removing some of her colleagues, while agent i would not be matched with anyone by inviting all her colleagues. □

We then present that the proposed algorithm asymptotically matches the upper bound presented in Theorem 7. While an approximation ratio $O(\frac{1}{n})$ is quite negative from the viewpoint of algorithm design, this is unavoidable as we require incentive compatibility property.

Proposition 2. *When the weights are restricted to* $\{-p, 0, p\}$ *for some constant* $p \in \mathbb{R}_+$*, algorithm 8 has an approximation ratio of* $\frac{1}{n}$ *when n is odd, and* $\frac{1}{n-1}$ *when n is even.*

Proof. Arbitrarily choose an SASHG-PS instance with n agents, and let m be the number of unmatched agents, i.e., the number of singleton coalitions, in the coalition structure π returned by the proposed algorithm. Since the proposed algorithm only constructs two-agent matchings as coalitions except for the singletons, there are $\frac{n-m}{2}$ matchings, and thus the social surplus is $\frac{n-m}{2}p$.

When m is either 0 or 1, it is obvious that adding an arbitrary edge to the same instance does not increase the number of two-agent matchings. By repeating this argument with adding a positive edge, we finally obtain the complete graph with n vertices as a utility graph, under which the optimal social surplus is $\frac{n(n-1)}{2}p$. Thus, the minimum ratio is $\frac{1}{n-1}$ for $m = 0$ and $\frac{1}{n}$ for $m = 1$, which coincides with the theorem statement.

Now we consider the case of $m \geq 2$. First, observe that there is no edge between any pair of agents who are left as singletons; otherwise the algorithm matches these agents, violating the property of maximum weight matching in the algorithm. Also, observe that the number of edges among the matched $n - m$ agents are at most $\frac{(n-m)(n-m-1)}{2}$, which corresponds to the case where these $n - m$ agents form a clique in the utility graph.

Now we count the total number of edges between those two sets of agents, namely the set M_1 of m singleton agents and the set M_2 of matched $n - m$ agents. For each pair $i, j \in M_2$ of agents who forms a two-agent coalition $\{i, j\} \in \pi$, the number of edges connecting $\{i, j\}$ and M_1 is at most m, all of which are connected to the exactly same vertices, say i; otherwise we can find an augmenting path, which also violates the property of maximum weight matching. Since there are $\frac{n-m}{2}$ matched pairs, the number of such edges is at most $\frac{m(n-m)}{2}$.

Therefore, the number of edges in the utility graph is at most

$$\frac{(n-m)(n-m-1)}{2} + \frac{m(n-m)}{2} = \frac{(n-m)(n-1)}{2}.$$

Thus, the ratio of the number of matchings to this value is at least $\frac{1}{n-1}$, which coincides with the theorem statement. □

One might feel that, since the algorithm can directly observe the utility graph among the participating agents, constructing a grand coalition whenever all the edges have a positive weight achieves a better social surplus. However, such an algorithm must also return as large coalition as possible even with negative weights; otherwise some agents have incentive to remove their colleagues with which edges with a negative weight exist, violating e.g., D-EPIC. We strongly believe that there is no much improvement from the above proposed algorithm, even in the average-case performance.

8 CONCLUSIONS

In this paper we clarified under which problem restrictions an appropriate coalition structure generation algorithm exist. Restricting the structure of utility graph worked well; we find an optimal algorithm that also satisfies C-DSIC, the most demanding incentive property. On the other hand, while restricting the weights provided a slightly positive findings, it might be possible that under some other weight restriction we may find an optimal algorithm.

As future works, there still exist various possible extensions of general hedonic games with permission structures, including joint manipulations by a group of agents (also known as group-strategyproofness), non-obvious manipulability, and randomized decision making. Also, some combinations of desirable properties introduced in this paper have not yet completely clarified, along with those incentive compatibility properties. Furthermore, it would be promising to clarify the key structure in the utility graphs that is essential to make the design of incentive compatible algorithms difficult.

REFERENCES

Ando, R., Kimura, K., Todo, T., and Yokoo, M. (2024). Parameterized voter relevance in facility location games with tree-shaped invitation graphs.

Ando, R., Kimura, K., Todo, T., and Yokoo, M. (2025). Parameterized voter relevance in facility location games with tree-shaped invitation graphs. To appear in *Proceedings of the 19th International Conference and Workshops on Algorithms and Computation (WALCOM '25)*.

Aziz, H. and Savani, R. (2016). *Hedonic Games*, pages 356–376. In *Handbook of Computational Social Choice*, chapter 15. Cambridge University Press.

Bogomolnaia, A. and Jackson, M. (2002). The stability of hedonic coalition structures. *Games and Economic Behavior*, 38(2):201–230.

Burani, N. and Zwicker, W. S. (2003). Coalition formation games with separable preferences. *Mathematical Social Sciences*, 45(1):27–52.

Cho, S.-H., Todo, T., and Yokoo, M. (2022). Two-sided matching over social networks. In *Proceedings of the 31st International Joint Conference on Artificial Intelligence (IJCAI-ECAI '22)*, pages 186–193.

Dreze, J. and Greenberg, J. (1980). Hedonic coalitions: Optimality and stability. *Econometrica*, 48(4):987–1003.

Edmonds, J. (1965). Maximum matching and a polyhedron with 0,1-vertices. *Journal of Research of the National Bureau of Standards Section B Mathematics and Mathematical Physics*, page 125.

Flammini, M., Kodric, B., Monaco, G., and Zhang, Q. (2021). Strategyproof mechanisms for additively separable and fractional hedonic games. *Journal of Artificial Intelligence Research*, 70:1253–1279.

Gairing, M. and Savani, R. (2019). Computing stable outcomes in symmetric additively separable hedonic games. *Mathematics of Operations Research*, 44(3):1101–1121.

Gilles, R. P., Owen, G., and van den Brink, R. (1992). Games with permission structures: The conjunctive approach. *International Journal of Game Theory*, 20(3):277–93.

Jeong, S. E. and Lee, J. (2024). The groupwise-pivotal referral auction: Core-selecting referral strategy-proof mechanism. *Games and Economic Behavior*, 143:191–203.

Kawasaki, T., Barrot, N., Takanashi, S., Todo, T., and Yokoo, M. (2020). Strategy-proof and non-wasteful multi-unit auction via social network. In *Proceedings of the 34th AAAI Conference on Artificial Intelligence (AAAI '20)*, pages 2062–2069.

Kawasaki, T., Wada, R., Todo, T., and Yokoo, M. (2021). Mechanism design for housing markets over social networks. In *Proceedings of the 20th International Conference on Autonomous Agents and Multiagent Systems (AAMAS '21)*, pages 692–700.

Li, B., Hao, D., and Zhao, D. (2020). Incentive-compatible diffusion auctions. In *Proceedings of the 29th International Joint Conference on Artificial Intelligence (IJCAI '20)*, pages 231–237. Main track.

Li, B., Hao, D., Zhao, D., and Zhou, T. (2017). Mechanism design in social networks. In *Proceedings of the 31st AAAI Conference on Artificial Intelligence (AAAI '17)*, pages 586–592.

van den Brink, R. (1997). An Axiomatization of the Disjunctive Permission Value for Games with a Permission Structure. *International Journal of Game Theory*, 26(1):27–43.

van den Brink, R. (2017). Games with a permission structure - A survey on generalizations and applications. *TOP: An Official Journal of the Spanish Society of Statistics and Operations Research*, 25(1):1–33.

You, B., Dierks, L., Todo, T., Li, M., and Yokoo, M. (2022). Strategy-proof house allocation with existing tenants over social networks. In *Proceedings of the 21st International Conference on Autonomous Agents and Multiagent Systems (AAMAS '22)*, pages 1446–1454.

Zhang, W., Zhao, D., and Chen, H. (2020). Redistribution mechanism on networks. In *Proceedings of the 19th International Conference on Autonomous Agents and Multiagent Systems (AAMAS '20)*, pages 1620–1628.

Zhao, D., Li, B., Xu, J., Hao, D., and Jennings, N. R. (2018). Selling multiple items via social networks. In *Proceedings of the 17th International Conference on Autonomous Agents and MultiAgent Systems (AAMAS '18)*, pages 68–76.