SURFACE-SURFACE INTERSECTION BY HERMITE INTERPOLATION

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Abstract: A fast heuristic to approximate the intersection curve of two surface patches was originally proposed by Sederberg and Nishita. The patches are rationally parametrized and they cut each other transversely. This paper reports a simple generalization that greatly improves the accuracy of the original heuristic. The generalization either avoids or attenuates the approximation error. Avoidance is achieved when the improved heuristic produces the exact intersection curve; attenuation is accomplished with an aggregate square distance formula guiding the selection of a generalized constraint for a better fit.

1 INTRODUCTION

Finding the intersection curve of two surfaces is a fundamental operation in computer graphics and computer geometric design and processing (Song et al., 2004). It is an intrinsically difficult problem and expectedly solutions depend on the surface description. In the realm of rationally parametrized surfaces, this difficulty is apparent because the intersection curve degree can be very high even when the surface degrees are modest. Recall that the degree of a curve in 3-space is the number of times it intersects a general plane. By Bezout's theorem (Cox et al., 1998), the degree of the intersection curve of two general tensor product surfaces of bi-degrees $m_1 \times m_2$ and $n_1 \times n_2$ is $4m_1m_2n_1n_2$, and that of two general triangular surfaces of total degrees m and n is m^2n^2. For the popular bi-cubic surface patches in computer geometry, the intersection curve degree works out to be 324! What is worst, the intersection curve of two rational surfaces need not be rational. It is well known that an algebraic curve is rational if and only if its genus is zero (Fulton, 1969). Katz and Sederberg (Katz and Sederberg, 1988) show that the genus g of the intersection curve of two general tensor product surfaces of parametric bi-degrees $m_1 \times m_2$ and $n_1 \times n_2$ is

$$g = 8m_1m_2n_1n_2 - 2m_1m_2(n_1+n_2) - 2n_1n_2(m_1+m_2)+1; \quad (1)$$

and that of two general triangular surfaces of parametric total degrees m and n is

$$g = 2m^2n^2 - \frac{3}{2}m^2n - \frac{3}{2}n^2m + 1. \quad (2)$$

Thus in general the intersection curve of two bi-degree surfaces is non-rational and that of two triangular surfaces is rational if and only if one is a plane and the other is either a plane or a quadric surface.

Currently, the many existing methods for finding the intersection curve of two surfaces may be broadly classified under either geometric subdivision or topological marching (de Figueiredo, 1996). Both approaches are computationally expensive if the solution is to be within certain given tolerance. With this state of affairs, perhaps the research on finding the intersection curve of two rationally parametrized surfaces should be to improve either the speed of "exact" solutions or the accuracy of approximate solutions. This paper aims the latter: its sole goal is to boost the accuracy of an approximate intersection curve as much as possible while incurring little additional computing costs.

The remainder of the paper comprises five sections. Section 2 reviews the original Sederberg-Nishita surface-surface intersection heuristic which relies on a constraint to solve for end-point tangents. Section 3 describes in detail the process and outcome of generalizing the original constraint deployed in the

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heuristic for better accuracy. Section 4 derives a formula to calculate some aggregate square distance and illustrates its use in finding a better approximate intersection curve. Section 5 discusses how the generalized constraint and the aggregate square distance can be used in tandem to enhance accuracy. Some issues that arise from this strategy are then discussed. Section 6 summarizes the paper.

2 THE ORIGINAL SEDERBERG-NISHITA HEURISTIC AND CONSTRAINT

The Sederberg-Nishita heuristic (Sederberg and Nishita, 1991) quickly approximates the intersection curve of two rationally parametrized surface patches that cut transversely between two known points. This situation may seem contrived but actually happens naturally when subdivision is employed to find the intersection of two surfaces. Indeed by subdividing, we transform the problem of finding the intersection of two larger surface patches to the problem of finding the intersections of a set of pairs of much smaller surface patches such that each pair intersect transversely. The intersection of the original surface patches is then the concatenation of these smaller intersections with G^1 continuity (Sederberg and Nishita, 1991).

2.1 The Problem Statement

The problem solved by the Sederberg-Nishita heuristic can be formulated precisely as follows. Two surface patches in the real Euclidean space $\mathbf{R}^3$ are given rational parametrically as

$$P : [0,1]\times[0,1] \to \mathbf{R}^3, \tag{3}$$

$$Q : [0,1]\times[0,1] \to \mathbf{R}^3. \tag{4}$$

Surfaces $P(s,t)$ and $Q(u,v)$ intersect transversely at two diagonally opposite points V_0 and V_1 where

$$V_0 = P(0,0) = Q(0,0),\ V_1 = P(1,1) = Q(1,1). \tag{5}$$

The task is to find the intersection curve

$$C = P(s,t) \cap Q(u,v), \quad 0 \le s,t,u,v \le 1. \tag{6}$$

2.2 Cubic Hermite Interpolation

The heuristic approximates the intersection curve using cubic Hermite interpolation. That is, given two end-point positions V_0 and V_1 and the corresponding tangents T_0 and T_1 of a curve, the cubic Hermite interpolation of the given curve is the cubic Bezier curve:

$$H(a) = \sum_{i=0}^{3} V_{i/3} B_i(a),\ \ 0 \le a \le 1, \tag{7}$$

where

$$V_{1/3} = V_0 + \frac{T_0}{3}, \quad V_{2/3} = V_1 - \frac{T_1}{3}; \tag{8}$$

and $B_i(a)$, $i = 0,1,2,3$, are the cubic Bernstein basis functions:

$$B_i(a) = \binom{3}{i}(1-a)^{3-i}a^i. \tag{9}$$

Since V_0 and V_1 are known, the heuristic amounts to finding the end-point tangents T_0 and T_1.

2.3 The Algorithm

The heuristic determines the end-point tangents T_0 and T_1 with the following steps.

1. Along the intersection curve, the parameters s, t, u, and v are parametrized with another parameter a such that

   $$s = s(a),\ \ t = t(a),\ \ u = u(a),\ \ v = v(a), \tag{10}$$

   and at $a = 0,1$, we have

   $$s(a) = t(a) = u(a) = v(a) = a. \tag{11}$$

2. Since $P(s(a),t(a)) = Q(u(a),v(a)), 0 \le a \le 1$, we have

   $$\frac{dP(s(a),t(a))}{da} = \frac{dQ(u(a),v(a))}{da}, 0 \le a \le 1. \tag{12}$$

3. The multivariate chain rule gives

   $$P_s s' + P_t t' = Q_u u' + Q_v v', \tag{13}$$

   where P_s, P_t, Q_u, Q_v are the respective partial derivatives and s', t', u', v' are derivatives with respect to a.

4. Linear algebra solves the above as ratios:

   $$\frac{s'}{|P_t Q_u Q_v|} = \frac{-t'}{|P_s Q_u Q_v|} = \frac{-u'}{|P_s P_t Q_v|} = \frac{v'}{|P_s P_t Q_u|}, \tag{14}$$

   where we treat P_s, P_t, Q_u, Q_v as 3×1 column vectors and the denominators are then 3×3 determinants. Note that we are interested in the two sets of ratios corresponding to $a = 0,1$.

5. To obtain definitive values for s', t', u', and v' when $a = 0, 1$, Sederberg and Nishita heuristically propose the constraint

$$s'(0) + t'(0) = s'(1) + t'(1) = 2. \quad (15)$$

6. With this Sederberg-Nishita constraint, we can solve for the parametric tangents $s'(a)$, $\cdots$, $v'(a)$ at $a = 0, 1$, and thus obtain the tangents T_a at positions V_a as

$$\begin{aligned} T_a &= P_{s(a)} s'(a) + P_{t(a)} t'(a), & (16) \\ &= Q_{u(a)} u'(a) + Q_{v(a)} v'(a), & (17) \end{aligned}$$

for $a = 0, 1$.

2.4 An Illustration

We illustrate the Sederberg-Nishita heuristic and constraint with the simplest example: $P(s,t)$ and $Q(u,v)$ are planes intersecting transversely from V_0 to V_1. The intersection curve is plainly

$$C(a) = (1-a)V_0 + aV_1, \quad 0 \le a \le 1. \quad (18)$$

Very pleasantly the cubic Hermite interpolation curve (7) constructed by the Sederberg-Nishita heuristic gracefully degenerates to become (18). That is, $H(a) = C(a)$. To see this, let

$$\begin{aligned} P(s,t) &= V_0 + Ss + Tt, \ 0 \le s,t \le 1, & (19) \\ Q(u,v) &= V_0 + Uu + Vv, \ 0 \le u,v \le 1, & (20) \end{aligned}$$

with

$$V_1 = V_0 + S + T = V_0 + U + V. \quad (21)$$

Since the partial derivatives

$$P_s, P_t, Q_u, Q_v = S, T, U, V, \quad (22)$$

are constants, ratios (14) at $a = 0, 1$ are the same:

$$\frac{s'}{|TUV|} = \frac{-t'}{|SUV|} = \frac{-u'}{|STV|} = \frac{v'}{|STU|}. \quad (23)$$

By (21) we have

$$|TUV| = -|SUV| \quad (24)$$

Consequently the parametric tangents are

$$\begin{aligned} s'(0) = t'(0) &= 1, & (25) \\ s'(1) = t'(1) &= 1, & (26) \end{aligned}$$

and the end-point tangents are

$$T_0 = T_1 = S + T = U + V = V_1 - V_0. \quad (27)$$

It is an exciting surprise that despite the use of a cubic curve (7) to approximate a line (18), not only the approximation turns out to be exact, but also the parametrization is proper rather than improper with index 3 (Sederberg, 1986).

Figure 1 shows two planes intersecting transversely and their intersection line.

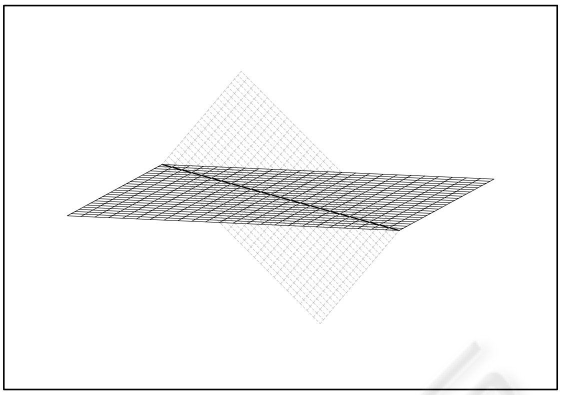

Figure 1: Two planar patches intersecting transversely and their intersection line.

3 GENERALIZING THE ORIGINAL HEURISTIC CONSTRAINT

Immediately we see that instead of imposing constraint (15) on $P(s,t)$, we could have imposed it on $Q(u,v)$. That is, we demand instead

$$u'(0) + v'(0) = u'(1) + v'(1) = 2. \quad (28)$$

After all, it is completely arbitrary which of the two given surface patches is called $P(s,t)$ and the other called $Q(u,v)$.

This observation motivates the search for a more symmetric constraint for determining the parametric tangents $s'(a)$, $t'(a)$, $u'(a)$, and $v'(a)$ at $a = 0, 1$.

3.1 Other Probable Linear Constraints

To explore the possibilities, we examine condition (11) carefully and see that it leads to

$$\begin{aligned} s(0) + t(0) + u(0) + v(0) &= 0, & (29) \\ s(1) + t(1) + u(1) + v(1) &= 4. & (30) \end{aligned}$$

These relations suggest we consider the constraint

$$\begin{aligned} &\sigma s(a) + \tau t(a) + \mu u(a) + \nu v(a) \\ &\quad = (\sigma + \tau + \mu + \nu) a, \quad (31) \end{aligned}$$

or equivalently,

$$\begin{aligned} &\sigma s'(a) + \tau t'(a) + \mu u'(a) + \nu v'(a) \\ &\quad = \sigma + \tau + \mu + \nu. \quad (32) \end{aligned}$$

Clearly constraint (32) is a generalization of the original constraint (15) because the latter is a special case of the former with

$$\sigma = \tau = 1, \ \mu = \nu = 0. \quad (33)$$

Consequently, a constraint completely symmetric to the patches P, Q and the parameters s, t, u, v would be

$$\sigma = \tau = \mu = \nu = 1. \quad (34)$$

It would be most desirable if being symmetric would guarantee the best approximation. Unfortunately, this is not to be so as we shall present a counter example in Section 4.

The preceding sad remark and the seemingly trivial derivation may cast doubts on the significance of the generalization. We shall restore confidence by demonstrating the usefulness of the generalized constraint in the following illustration and later discussions. Most significantly, we show in some situations the generalization produces the exact intersection with appropriate σ, τ, μ, and ν.

3.2 Illustrating the Generalized Constraint

The following shows that a twisted cubic curve, which is the intersection of a quadric cylinder and a cubic cylinder, can be obtained exactly using the generalized constraint (32) with appropriate σ, τ, μ, ν.

Let the quadric cylinder $\alpha^2 y = \beta x^2$ and the cubic cylinder $\alpha^3 z = \gamma x^3$ be parametrized respectively as

$$P(s,t) = (\alpha s, \beta s^2, \gamma t),\ 0 \le s,t \le 1; \quad (35)$$
$$Q(u,v) = (\alpha u, \beta v, \gamma u^3),\ 0 \le u,v \le 1. \quad (36)$$

Equivalently,

$$P(s,t) = As + Bs^2 + Ct, \quad (37)$$
$$Q(u,v) = Au + Bv + Cu^3. \quad (38)$$

where

$$A = (\alpha,0,0), B = (0,\beta,0), C = (0,0,\gamma). \quad (39)$$

The exact intersection $C(a)$ from $V_0 = 0$ to $V_1 = A + B + C$ is easily solved analytically and can be parametrized as

$$C(a) = (\alpha a, \beta a^2, \gamma a^3),\ \ 0 \le a \le 1. \quad (40)$$

To find an approximation intersection curve $H(a)$ by the heuristic we first compute

$$P_s = A + 2Bs, \qquad P_t = C, \quad (41)$$
$$Q_u = A + 3Cu^2, \qquad Q_v = B. \quad (42)$$

The parametric tangent ratio is

$$\begin{aligned} & s'(a) : t'(a) : u'(a) : v'(a) \\ = \ & |C, A + 3Cu^2, B| \\ : \ & -|A + 2Bs, A + 3Cu^2, B| \\ : \ & -|A + 2Bs, C, B| \\ : \ & |A + 2Bs, C, A + 3Cu^2| \\ = \ & 1 : 3u^2 : 1 : 2s \end{aligned} \quad (43)$$

To apply the original constraint on $P(s,t)$, we set $\sigma = \tau = 1$, $\mu = \nu = 0$ and obtain parametric tangents

$$s'(0), t'(0), u'(0), v'(0) = 2, 0, 2, 0; \quad (44)$$
$$s'(1), t'(1), u'(1), v'(1) = \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, 1; \quad (45)$$

and end-point tangents

$$T_0 = 2A,\ \ T_1 = \frac{A + 2B + 3C}{2}. \quad (46)$$

The resulting approximation cubic Hermite curve is

$$\frac{a}{2}\left(a^2 - 3a + 4\right) A + a^2 (2 - a) B + \frac{a^2}{2}(3 - a) C. \quad (47)$$

To apply the original constraint on $Q(u,v)$, we set $\sigma = \tau = 0$, $\mu = \nu = 1$ and obtain parametric tangents

$$s'(0), t'(0), u'(0), v'(0) = 2, 0, 2, 0; \quad (48)$$
$$s'(1), t'(1), u'(1), v'(1) = \frac{2}{3}, 2, \frac{2}{3}, \frac{4}{3}; \quad (49)$$

and end-point tangents

$$T_0 = 2A,\ \ T_1 = \frac{2A + 4B + 6C}{3}. \quad (50)$$

The resulting approximation cubic Hermite curve is

$$\frac{a}{3}\left(2a^2 - 5a + 6\right) A + \frac{a^2}{3}(5 - 2a) B + a^2 C. \quad (51)$$

In both the above cases the original constraint did not produce the exact intersection. But observe that the intersection is found by setting

$$s = a, t = a^3, u = a, v = a^2. \quad (52)$$

Thus we may simply set either σ or μ to 1 and the rest to zero to produce the exact intersection (40). Either way, the parametric tangents are

$$s'(0), t'(0), u'(0), v'(0) = 1, 0, 1, 0; \quad (53)$$
$$s'(1), t'(1), u'(1), v'(1) = 1, 3, 1, 2; \quad (54)$$

and the end-point tangents are

$$T_0 = A,\ \ T_1 = A + 2B + 3C. \quad (55)$$

This example reveals that the original constraint may not produce the exact intersection even when it is a cubic curve. In contrast, the generalized constraint is able to produce the exact intersection.

It is worth mentioning that this example does not contradict Bezout's theorem despite that the degree of the intersection curve of a degree two and a degree three surfaces is three but not six. This is because here the quadric cylinder and the cubic cylinder has another intersection line $x = w = 0$ at infinity of multiplicity three. (Here w is the homogenizing coordinate to enlarge the affine 3-space to a projective 3-space.)

Figure 2 shows the intersection curve, a twisted cubic curve, and the transversely intersecting quadric cylinder patch and cubic cylinder patch. Figure 3 shows the exact intersection curve (with $\sigma = 1$, $\tau = \mu = \nu = 0$ or $\mu = 1$, $\sigma = \tau = \nu = 0$) and the two approximation intersection curves obtained with $\sigma = \tau = 1$, $\mu = \nu = 0$ and $\sigma = \tau = 0$, $\mu = \nu = 1$.

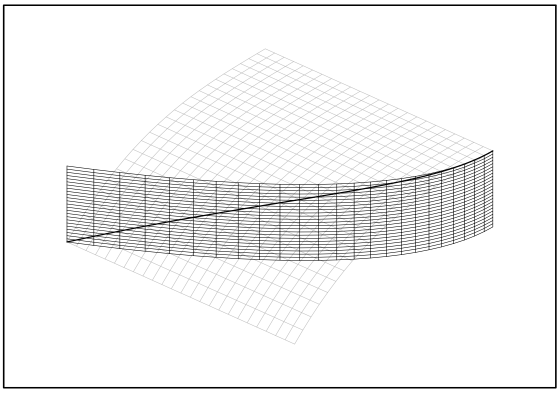

Figure 2: The intersection curve of a quadric cylinder patch and a cubic cylinder patch.

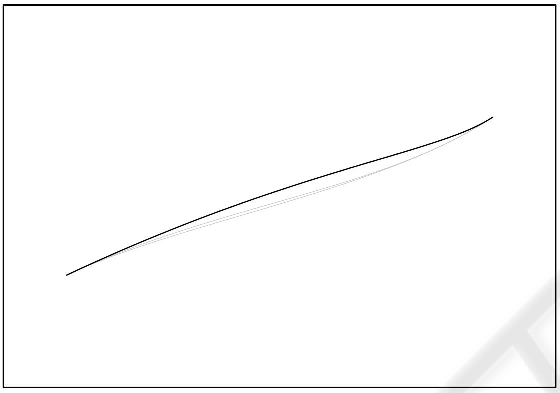

Figure 3: The two approximation intersection curves (in thin line) and the exact intersection curve (in thick line) of a quadric cylinder patch and a cubic cylinder patch.

4 ESTIMATING THE CLOSENESS OF FIT

By now we see that there are infinitely many generalized constraints for determining the parametric tangents $s'(a)$, $t'(a)$, $u'(a)$, and $v'(a)$ at $a = 0,1$. This calls for some criteria to assess among constraints κ which gives a better or best Hermite interpolation $H_\kappa(a)$ to approximate the exact intersection $P(s,t) \cap Q(u,v)$, ideally parametrized as $C(a)$, $0 \le a \le 1$. For this purpose we propose a closeness of fit measurement called *aggregate square distance*.

4.1 The Aggregate Square Distance

Let $(s(a),t(a))$, $(u(a),v(a))$ be the pre-images of the intersection curve $C(a)$, $0 \le a \le 1$, in the parameter spaces (s,t), (u,v) respectively. That is,

$$(s(a),t(a)) = P^{-1}(C(a)), \quad (56)$$

$$(u(a),v(a)) = Q^{-1}(C(a)), \quad (57)$$

where $0 \le a \le 1$. Let $|V|$ be the usual Euclidean norm for the 1×3 vector $V \in \mathbf{R}^3$; that is,

$$|(x,y,z)| = \sqrt{x^2+y^2+z^2}. \quad (58)$$

Consider the integral

$$I = \int_0^1 |P(s(a),t(a)) - Q(u(a),v(a))|^2 da. \quad (59)$$

It is obvious that $I = 0$ if and only if $P(s(a),t(a)) = Q(u(a),v(a))$. That is, the integral vanishes if and only if $P(s(a),t(a))$ and $Q(u(a),v(a))$ give the exact intersection curve $C(a)$.

However, if $(s(a),t(a))$ and $(u(a),v(a))$ are only approximations of the pre-images of the actual intersection curve $C(a)$, the integral will give the aggregate square distance of the two approximation curves $C_P(a) = P(s(a),t(a))$ and $C_Q(a) = Q(u(a),v(a))$. In other words, the integral is a measurement of how close the two approximation curves $C_P(a)$ and $C_Q(a)$ are. It is thus reasonable to extrapolate it to be a measurement of how well $H(a)$ fits $C(a)$. Admittedly this is a heuristic estimation of the closeness of fit, but empirically it seems to reflect the closeness of fit quite well as shall be illustrated later.

After adopting the aggregate square distance as a measurement for the closeness of fit of $H(a)$ to $C(a)$, our remaining task is to find some approximation pre-images $(s(a),t(a))$ and $(u(a),v(a))$ of $C(a)$ with as little computing costs as possible.

4.2 Approximating the Pre-Images

Recall that we have already found the parametric tangents $s'(a)$, $t'(a)$, $u'(a)$, and $v'(a)$ for $a = 0,1$. It is thus with no additional computing costs if we approximate the pre-image of the intersection curve $C(a)$ in the parameter space (s,t) also with a Hermite cubic interpolation $H_P(a)$, by setting the control vertices in (7) as

$$V_0 = (0,0), \quad (60)$$

$$V_{1/3} = V_0 + \frac{(s'(0),t'(0))}{3}, \quad (61)$$

$$V_{2/3} = V_1 - \frac{(s'(1),t'(1))}{3}, \quad (62)$$

$$V_1 = (1,1). \quad (63)$$

The approximation pre-image $H_Q(a)$ of the intersection curve $C(a)$ in the (u,v) space is constructed similarly with $(s'(a),t'(a))$ replaced by $(u'(a),v'(a))$ for $a = 0,1$.

To summarize, the intersection curve $C(a)$ and its pre-images are approximated by three approximation curves, one in the Euclidean 3-space (x,y,z) and two in the parameter 2-spaces (s,t) and (u,v):

$$C(a) = P(s,t) \cap Q(u,v) \approx H(a), \quad (64)$$

$$P^{-1}(C(a)) \approx H_P(a), \quad (65)$$

$$Q^{-1}(C(a)) \approx H_Q(a). \quad (66)$$

Furthermore, we estimate

$$\int_0^1 |H(a) - C(a)|^2 da \quad (67)$$

using

$$\int_0^1 |P(H_P(a)) - Q(H_Q(a))|^2 da. \quad (68)$$

The following theorem affirms the relevance of integral (68) as a measurement of closeness of fit.

Theorem 1 *If integral (68) vanishes and $C(a)$ is cubic in a, then $H(a) = C(a)$.*

Proof.

Integral (68) vanishes if and only if $P(H_P(a)) = Q(H_Q(a))$. But the equality means

$$P(H_P(a)) = Q(H_Q(a)) = P(s,t) \cap Q(u,v) = C(a).$$

Thus $C(a)$ and $H(a)$ have the same end-points and end-point tangents. Consequently they give the same curve since a parametric cubic curve over $[0,1]$ is completely determined by its end-points and end-point tangents. Q.E.D.

We emphasize that $H_P(a)$ and $H_Q(a)$ are by-products from the construction of $H(a)$. It is thus fair to claim they are available free. Integral (68) can be integrated numerically with high efficiency because the integrands are polynomials of low degrees.

4.3 Illustrations

We repeat the tutorial example in (Sederberg and Nishita, 1991) to illustrate the role of the aggregate square distance in identifying a constraint for a closer-fitting approximation $H(a)$.

Consider two bilinear patches

$$P(s,t) = \sum_{i,j=0}^{1} P_{ij}(1-s)^{1-i}s^i(1-t)^{1-j}t^j, \quad (69)$$

and

$$Q(u,v) = \sum_{i,j=0}^{1} Q_{ij}(1-u)^{1-i}u^i(1-v)^{1-j}v^j, \quad (70)$$

where

$$P_{00}, P_{10}, P_{01}, P_{11} = [0,0,0], [0,1,4], [3,3,0], [4,0,4]; \quad (71)$$

and

$$Q_{00}, Q_{10}, Q_{01}, Q_{11} = [0,0,0], [0,4,4], [4,2,0], [4,0,4]. \quad (72)$$

The example in (Sederberg and Nishita, 1991) used the Sederberg-Nishita constraint on $P(s,t)$ by setting

$$\sigma = \tau = 1, \ \mu = \nu = 0, \quad (73)$$

and obtained the parametric tangents

$$s'(0), t'(0), u'(0), v'(0) = \frac{2}{3}, \frac{4}{3}, \frac{2}{3}, 1; \quad (74)$$

$$s'(1), t'(1), u'(1), v'(1) = 2, 0, 2, \frac{1}{2}. \quad (75)$$

The resulting end-point tangents were

$$T_0 = \left[4, \frac{14}{3}, \frac{8}{3}\right], T_1 = [2, -6, 8], \quad (76)$$

and thus

$$H_{1100}(a) = \begin{bmatrix} -2a(a^2-a-2) \\ -\frac{2a}{3}(2a^2+5a-7) \\ \frac{4a}{3}(2a^2-a+2) \end{bmatrix}^T. \quad (77)$$

The aggregate square distance for this approximation is

$$\rho_{1100} = 0.0053561. \quad (78)$$

Now we try using the Sederberg-Nishita constraint on $Q(u,v)$ instead of $P(s,t)$; that is, we set

$$\sigma = \tau = 0, \ \mu = \nu = 1. \quad (79)$$

The parametric tangents are

$$s'(0), t'(0), u'(0), v'(0) = \frac{4}{5}, \frac{8}{5}, \frac{4}{5}, \frac{6}{5}; \quad (80)$$

$$s'(1), t'(1), u'(1), v'(1) = \frac{8}{5}, 0, \frac{8}{5}, \frac{2}{5}. \quad (81)$$

The resulting end-point tangents are

$$T_0 = \left[\frac{24}{5}, \frac{28}{5}, \frac{16}{5}\right], T_1 = \left[\frac{8}{5}, -\frac{24}{5}, \frac{32}{5}\right], \quad (82)$$

and thus

$$H_{0011}(a) = \frac{4a}{5}\begin{bmatrix} -(2a^2-a-6) \\ (a^2-8a+7) \\ (2a^2-a+4) \end{bmatrix}^T. \quad (83)$$

The aggregate square distance for this approximation is

$$\rho_{0011} = 0.00032996. \quad (84)$$

Since ρ_{0011} is much smaller than ρ_{1100}, we expect $H_{0011}(a)$ to be a closer fit than $H_{1100}(a)$. This is found to be so by visually inspecting a plot of $P(s,t)$, $Q(u,v)$, $H_{1100}(a)$, and $H_{0011}(a)$.

Finally we assess the performance of the symmetric constraint

$$\sigma = \tau = \mu = \nu = 1. \tag{85}$$

The parametric tangents are

$$s'(0), t'(0), u'(0), v'(0) = \frac{8}{11}, \frac{16}{11}, \frac{8}{11}, \frac{12}{11}; \tag{86}$$

$$s'(1), t'(1), u'(1), v'(1) = \frac{16}{9}, 0, \frac{16}{9}, \frac{4}{9}. \tag{87}$$

The resulting end-point tangents are

$$T_0 = \left[\frac{48}{11}, \frac{56}{11}, \frac{32}{11}\right], T_1 = \left[\frac{16}{9}, -\frac{16}{3}, \frac{64}{9}\right], \tag{88}$$

and thus

$$H_{1111}(a) = \frac{4a}{99}\begin{bmatrix} -(46a^2 - 37a - 108) \\ -6(a^2 + 20a - 21) \\ (50a^2 - 23a + 72) \end{bmatrix}^T. \tag{89}$$

The aggregate square distance for this approximation is

$$\rho_{1111} = 0.0016125. \tag{90}$$

We observe that

$$\rho_{0011} < \rho_{1111} < \rho_{1100}. \tag{91}$$

With a visual inspection of a plot of $P(s,t)$, $Q(u,v)$, $H_{0011}(a)$, $H_{1111}(a)$, and $H_{1100}(a)$, it is satisfying to see that the aggregate square distance does faithfully reflect the closeness of fit.

Figure 4 shows the bilinear patches and their intersection curve. Figure 5 and figure 6 shows the three approximation intersection curves $H_{1100}(a)$, $H_{0011}(a)$, $H_{1111}(a)$ and the actual intersection curve $C(a)$. Visually we ascertain that $H_{0011}(a)$ fits best while $H_{1100}(a)$ fits worst and $H_{1111}(a)$ is somewhere in between. This is consistent with the prediction of their aggregate square distances.

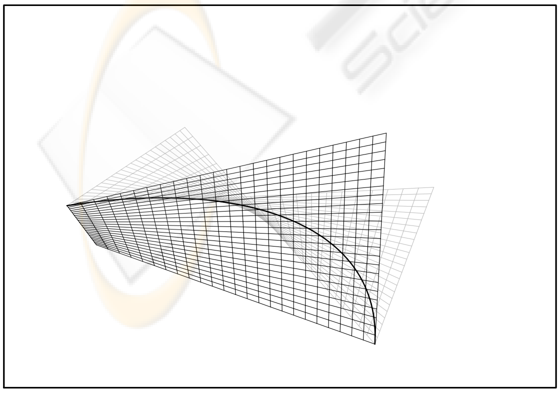

Figure 4: The bilinear patches $P(s,t)$, $Q(u,v)$, and their intersection $C(a)$.

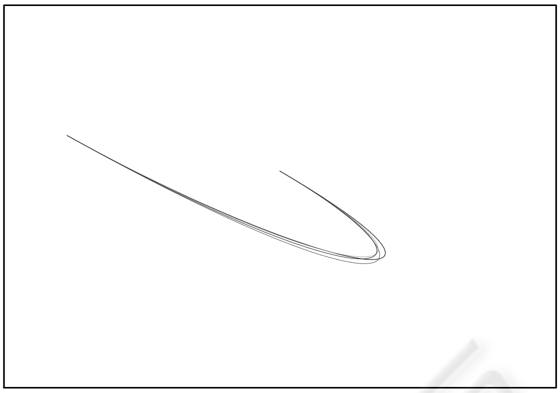

Figure 5: The cubic Hermite curves $H_{1100}(a)$, $H_{1111}(a)$, $H_{0011}(a)$ and the exact intersection curve $C(a)$, $0 \leq a \leq 1$, for the bilinear patches.

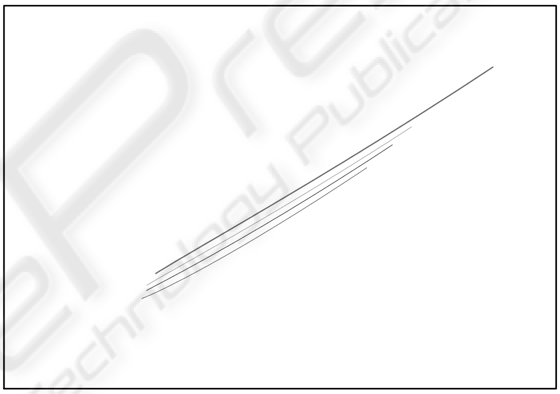

Figure 6: The top curve is the exact intersection curve. In increasing distance from the top curve are the curves obtained with constraints $\sigma = \tau = 0, \mu = \nu = 1$; $\sigma = \tau = \mu = \nu = 1$; $\sigma = \tau = 1, \mu = \nu = 0$ respectively. All curves are plotted from 0.3 to 0.7 to magnify their separation.

5 OBSERVATIONS AND SPECULATIONS

From the above discussions, we realize that the approximation cubic Hermite curves $H_{1100}(a)$ and $H_{0011}(a)$ fit the intersection curve $C(a)$ with different accuracy and one can be a much better fit than the other. The winner can be determined by comparing their aggregate square distances ρ_{1100} and ρ_{0011}, which can be computed efficiently with little additional computing costs.

Intuitively, we would expect the approximation cubic Hermite curve $H_{1111}(a)$, in terms of accuracy, to be worst than the best of $H_{1100}(a)$, $H_{0011}(a)$ but better than the worst of them. This speculation is supported by our example. It would be very helpful in establishing the applicability of the aggregate square distance measurement if this guess can be established as a fact with some proof.

We have shown that when the surface patches are planar, the Sederberg-Nishita heuristic produces the exact intersection line with a proper parametrization, despite the use of a cubic Hermite interpolation. This has greatly enhanced the credibility of the Sederberg-Nishita constraint used in the heuristic. It would be interesting in theory and useful in practice if more cases can be discovered in which the cubic Hermite curve is the actual intersection curve rather than just an approximation. It would be even better if some sufficient or necessary conditions can be laid down under which the generalized constraint produces the exact intersection curve.

In this paper we only exploit polynomial cubic Hermite interpolation. It should be a worthwhile effort to explore the use of rational Hermite interpolation (Goldman, 2003) for approximating the intersection curve of two surface patches. Note that some rational curves, such as circular arcs, do not have polynomial parametrization.

6 CONCLUSIONS

We first reviewed a quick heuristic originally proposed by Sederberg and Nishita. The heuristic finds a cubic Hermite interpolation curve that approximates the intersection of two rationally parametrized surface patches when they intersect transversely. The power of the heuristic was established empirically in (Sederberg and Nishita, 1991). We further enhanced its credibility by showing that when the surfaces are planes the heuristic actually produces the exact intersection line parametrized properly.

The heuristic utilizes a constraint to decide the parametric tangents in order to compute the Hermite interpolation. The constraint is arbitrarily applied to one of the two surfaces in the heuristic. We proposed that the constraint should be applied to both surfaces individually thus producing two approximating cubic Hermite curves. The better-fitting one should then be used. The assessment is based on their aggregate square distances which can be easily evaluated. An example was given to illustrate that much improvement in accuracy could be achieved. All this was done with very little additional computing costs, as it was crucial not to sacrifice speed of the heuristic to attain better accuracy.

This way of applying the constraint can be considered as special cases of deploying of what we called the generalized constraint. We showed that there are situations in which the generalized constraint could produce the exact intersection curve but the original constrain could not.

We also discussed some interesting open problems with this new perspective on the constraint.

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