Uncertainty Measure of Process Models using Entropy and Petri Nets

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Abstract: In recent years, many measures of process models have been proposed to predict or quantify the number of specific properties. These properties may include readability, complexity, cohesion or uncertainty of process models. The content of this work is to propose a method that allows the measurement of uncertainty in the process models, which can be expressed in the form of a Petri net. The actual method works by mapping the set of all reachable marking of Petri net to Markov chain and subsequent quantification of steady-state probabilities of its states. Uncertainty is then quantified as the entropy of states in the Markov chain. Uncertainty can also be expressed as a percentage of the calculated entropy to the maximum entropy of a Petri net.

1 INTRODUCTION

Currently, there are a large number of different modelling languages, which serve to describe business processes. They differ in their notation, complexity, mathematical foundation and other characteristics. From a comparative perspective, in recent years, many measures of process models have been proposed that aims to create a variety of metrics for analysing process models in terms of complexity (Lassen and van der Aalst, 2009); (Rolón et al., 2009), uncertainty (Jung et al., 2011) or cohesion (Reijers and Vanderfeesten, 2004). These metrics are then used for various purposes, such as evaluation of user-friendliness, understandability, usability, maintainability and other (González et al., 2010). In the following is proposed the approach, which allow the analysis and quantification of the uncertainty of any process model, which has been modelled using the classic P/T (Place/Transition) Petri net, or can be remap into it (van der Aalst, 1998). Quantification of the uncertainty of any process model implies the predicted behaviour of the modelled process and therefore its degree of predictability. Reducing uncertainty in process models can lead to better predictability of process behaviour and also improve managerial efficiency.

Petri nets are a suitable tool for modelling discrete event dynamic systems which feature concurrency, parallelism and synchronization. Their main advantage is the ability to precisely verify the assumptions imposed on the model. Petri nets have been defined by Carl Adam Petri in 1962 (Petri, 1962) and since then, their development evolves in many directions. One direction is to define a new model features that extend the verification power of Petri nets. These are primarily properties of Petri nets such as liveness, boundedness, reachability and many more. Most of these properties also require a number of assumptions that restrict the definition of Petri net. Another direction of development is expanding definition of Petri nets by adding new elements to refine and simplify modelling (but mostly with a lower degree of formality). Examples are timed and stochastic Petri nets, which allow refining the individual state changes, taking into account time-consumption (deterministically or stochastically). Another example is coloured Petri nets, which combine the basic petri net with another modelling language, thus dramatically expanding (and mainly simplify) modelling capabilities Petri nets. The main drawback of this second direction is limited verification options.

The aim of this work is to define a method that allows quantifying the uncertainty of Petri net models. This objective is achieved using the concepts of information theory (Shannon´s entropy (Shannon, 1948)) and stochastic processes (Markov chains).

This work is divided into 5 sections. The second section presents the basic definition of Petri nets and

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other terms that relate to the issue. The third chapter presents the basic issues of Markov chains associated with the determination of steady-state probabilities. The fourth chapter contains a definition of Shannon entropy and the method of uncertainty calculation over arbitrary Petri net. The fifth section aims to illustrate the issues defined in the previous section on simple example. The sixth section discusses the advantages and disadvantages of the presented method. The last section concludes this paper and proposes possibilities for further expansion of this issue.

2 PETRI NETS

Currently, there are a number of basic definitions of petri nets, which are distinguished by its formality (restrictions and verification force). The following is the general definition of P/T (Place/Transition) Petri nets, which allows quantifying the edges (arcs) with positive integers.

Definition 2.1: Generalized P/T Petri net is a 5-tuple, $PN = (P, T, F, W, M_0)$ where:

- $P = \{p_1, p_2, p_{3,}, \dots, p_m\}$ – a finite set of places,
- $T = \{t_1, t_2, t_3, \dots, t_n\}$ – a finite set of transitions,
- $P \cap T = \emptyset$ – places and transitions are mutually disjoint sets,
- $F \subseteq (P \times T) \cup (T \times P)$ – a set of edges (arcs), defined as a subset of the set of all possible connections,
- $W: F \rightarrow N_1$ – a weight function, defines the multiplicity of edges (arcs),
- $M_0: P \rightarrow N_0$ – an initial marking.

Such a definition does not contain any implicit restriction in terms of capacity of individual places. If it is required to model capacity constraints on some subset of places it is possible to use the so-called complementary-place transformation to adjust net as required (Murata, 1989). In practice, it is advantageous to specify a priori capacity of individual sites and thus simplifying subsequent analysis of model (eliminates the problem of infinite capacity).

Definition 2.2: Capacity P/T Petri net is a 6-tuple, $PN = (P, T, F, W, C, M_0)$ where:

- $P = \{p_1, p_2, p_{3,}, \dots, p_m\}$ – a finite set of places,
- $T = \{t_1, t_2, t_3, \dots, t_n\}$ – a finite set of transitions,
- $P \cap T = \emptyset$ – places and transitions are mutually disjoint sets,
- $F \subseteq (P \times T) \cup (T \times P)$ – a set of edges, defined as a subset of the set of all possible connections,
- $W: F \rightarrow N_1$ – a weight function, defines the multiplicity of edges,
- $C : P \rightarrow N_1$ – capacities of places,
- $M_0: P \rightarrow N_0$ – an initial marking.

Definition 2.3: Marking of Petri net

Let $PN = (P, T, F, W, C, M_0)$ is a Petri net. Map $M: P \rightarrow N_0$, is called marking of Petri net PN.

Marking represents the state of the network after execution a specific number of steps, i.e. the firing a specific number of enabled transitions. If a transition is enabled (or not) depends on the net structure and the actual marking.

Definition 2.4: Pre-set, Post-set

Let $PN = (P, T, F, W, C, M_0)$ is a Petri net. Pre-sets and post-sets are defined as:

- ${}^\bullet p = \{t | (t, p) \in F\}$ – the set of input transitions of p,
- ${}^\bullet t = \{p | (p, t) \in F\}$ – the set of input places of t,
- $p^\bullet = \{t | (p, t) \in F\}$ – the set of output transitions of p,
- $t^\bullet = \{p | (t, p) \in F\}$ – the set of output places of t.

Definition 2.5: Enabled transition

Let $PN = (P, T, F, W, C, M_0)$ is a Petri net. Transition $t \in T$ is called enabled with marking M (M-enabled), if

$$\forall p \in {}^\bullet t: M(p) \geq W(p, t)$$

$$\forall p \in t^\bullet: M(p) \leq C(p) - W(t, p)$$

Definition 2.6: Next marking

Let $PN = (P, T, F, W, C, M_0)$ is a Petri net and M is its marking. If a transition $t \in T$ is enabled at marking M, then by its execution is obtained next marking M', which is defined as follows:

$$\forall p \in P: M'(p) = \\ = \begin{cases} M(p) - W(p, t), & if\ p \in {}^\bullet t \backslash t^\bullet \\ M(p) + W(t, p), & if\ p \in t^\bullet \backslash {}^\bullet t \\ M(p) - W(p, t) + W(t, p), & if\ p \in t^\bullet \cap {}^\bullet t \\ M(p) & otherwise \end{cases}$$

The situation that the transition t changes the marking M to M', is usually expressed as $M[t\rangle M'$.

Definition 2.7: Sequence of transitions, reachability

Let $PN = (P, T, F, W, C, M_0)$ is a Petri net. Sequence of transitions σ is the sequence of enabled transition that lead from marking M to another marking M'. This situation is denoted as $M[\sigma\rangle M'$. A marking for which there is a sequence of transitions from the initial marking is called reachable marking.

Definition 2.8: The set of all reachable marking

Let $PN = (P,T,F,W,C,M_0)$ is a Petri net and M is its marking. The set of all possible markings reachable from initial marking M_0 in a Petri net PN is denoted by $R(PN,M_0)$ or simply $R(M_0)$.

$$R(M_0) = \begin{bmatrix} M_0(p_1) & M_1(p_1) & \cdots & M_{|R(M_0|}(p_1) \\ M_0(p_2) & M_1(p_2) & \cdots & M_{|R(M_0|}(p_2) \\ \vdots & \vdots & \ddots & \vdots \\ M_0(p_m) & M_1(p_m) & \cdots & M_{|R(M_0|}(p_m) \end{bmatrix}$$

Definition 2.9: Boundedness

Let $PN = (P,T,F,W,C,M_0)$ is a Petri net. Place $p \in P$ is called k-bounded if:

$$\exists k \in N_1 : \forall M \in R(M_0) : M(p) \leq k$$

Place $p \in P$ is called bounded, if it is k-bounded for some $k \in N_1$. If every place in PN is bounded, then this net is called bounded Petri net.

Definition 2.10: Live marking, live net

Let $PN = (P,T,F,W,C,M_0)$ is Petri net. Marking $M \in R(M_0)$ is live, if $\forall t \in T$ exist some marking $M_1 \in R(M_0)$ such that transition t is M_1-enabled. If $\forall M \in R(M_0)$ is live, then PN is live.

3 PROBABILITY OF MARKINGS AND MARKOV CHAINS

The set of all reachable markings can be expressed in terms of Markov chains. For the purposes of defining the steady-state probability of each marking $M \in R(M_0)$ is necessary to define the transition matrix.

Definition 3.1: Transition matrix

Let $PN = (P,T,F,W,C,M_0)$ is a Petri net and $R(M_0)$ its reachability set. Transition matrix A of Petri net PN is defined as:

$$\boldsymbol{A}: \left(R(M_0) \times R(M_0)\right) \rightarrow <0,1>$$

Where values are made according following rule and the matrix A form right stochastic matrix:

$$\boldsymbol{A}_{i,j} = \begin{cases} 0 & \nexists(t \in T): M_i[t\rangle M_j \\ \dfrac{1}{|M_j|} & \exists(t \in T): M_i[t\rangle M_j \end{cases}$$

Where $|M_j|$ represents the number of marking that are reachable from M_i. In this way each branching in the state space (graph) assigned uniform probabilities between different paths. However, explicitly chosen values of probabilities for various branches can be used as well, subject to the condition $\sum_{j=1}^{|R(M_0)|} A_{i,j} = 1$ (right stochastic matrix).

Definition 3.2: Steady-state probabilities

Let $PN = (P,T,F,W,C,M_0)$ is a Petri net and $\boldsymbol{A}$ is its transition matrix. Steady-state distribution vector $\boldsymbol{u}$ is defined as left eigenvector of transition matrix $\boldsymbol{A}$:

$$\boldsymbol{uA} = \boldsymbol{u}$$

Vector $\boldsymbol{u}$ then represents the probabilities of all markings from $R(M_0)$:

$$\boldsymbol{u} = \begin{bmatrix} \Pr(M_0) \\ \Pr(M_1) \\ \vdots \\ \Pr(M_{|R(M_0)|}) \end{bmatrix}$$

Definition 3.3: Long term probability of marking $M \in R(M_0)$ is defined as a corresponding element of vector $\boldsymbol{u}$:

$$u_i = \Pr(M_i)$$

The probability of marking M can be seen as a joint probability of markings of individual places:

$$Pr(M) = \Pr(M(p_1) = x_1, M(p_2) = x_2, \dots, M(p_m) = x_m)$$

When examining the steady-state probabilities it is appropriate to place emphasis on liveness of analysing model, since each dead marking of Petri net corresponds to absorb state in terms of Markov chains. Each absorption state can always occur, i.e. its probability equal 1 and thus all live markings have probability equal 0. This would lead to a deterministic model without any uncertainty.

4 ENTROPY

Entropy can measure the amount of disorder, which is associated with a random variable.

Definition 4.1: The entropy of the random variable X is defined as:

$$H(X) = -\sum_x Pr(X = x) \log_2 Pr(X = x)$$

With the assumption $0 \cdot \log_2(0) \equiv 0$.

Definition 4.2: Joint entropy

The joint entropy of two discrete random variables X and Y is defined as

$$H(X,Y) = -\sum_x \sum_y Pr(X = x, Y = y) \log_2 Pr(X = x, Y = y)$$

Where x and y are particular values of X and Y, respectively $\Pr(X = x, Y = y)$ is the probability of these values occurring together and the general form for n random variables:

$$H(X_1, \ldots, X_n) = -\sum_{x_1} \ldots \sum_{x_n} Pr(X_1 = x_1, \ldots, X_n = x_n) \log_2 Pr(X_1 = x_1, \ldots, X_n = x_n)$$

Definition 4.3: Entropy of Petri net

Let $PN = (P, T, F, W, C, M_0)$ is a Petri net and $\boldsymbol{u}$ is a vector of steady-state probabilities $u_i = \Pr(M_i)$, $M_i \in R(M_0)$. Entropy of PN is defined as

$$H(PN) = -\sum_{i=1}^{|R(M_0)|} \eta_i \log_2 \eta_i$$

Definition 4.4: Uncertainty of Petri net

Let $PN = (P, T, F, W, C, M_0)$ is a Petri net and $H(PN)$ its entropy. Uncertainty of PN is defined as

$$Uncertainty(PN) = \frac{H(PN)}{\log_2 |R(M_0)|}$$

Uncertainty value is then located in the interval <0,1>, where 0 stands for fully deterministic model and 1for absolute chaotic model. The more is the uncertainty value close to 1, the less is predictable the behaviour of the model.

5 EXAMPLE OF SIMPLE MODEL

As a simple example, consider a Petri net, which is composed of 5 places and 5 transitions, see Figure 1. The model contains some typical elements that are abundant in classic process models. These elements are for instance sequence (transition T4), AND-split (transition T1), XOR (transition T2 and T3) and cycle (transition T5). For more information on the mapping of these (and other) elements into Petri net can be found in (Jung et al., 2011).

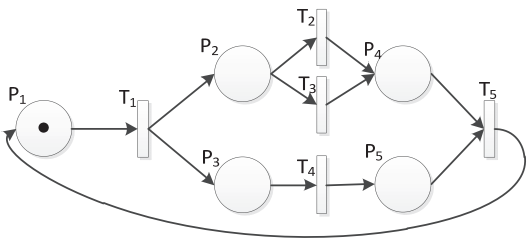

Figure 1: Petri net example.

The set of all reachable markings $R(M_0)$ of the Petri net contains five markings:

	M_0	M_1	M_2	M_3	M_4
p_1	1	0	0	0	0
p_2	0	1	0	1	0
p_3	0	1	1	0	0
p_4	0	0	1	0	1
p_5	0	0	0	1	1

The corresponding state space (graph) is shown in Figure 2.

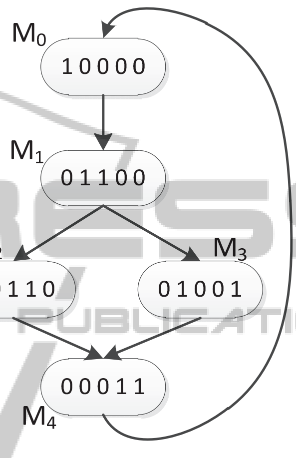

Figure 2: State space.

This state space corresponds to Markov chain, which generates the following transition matrix:

	M_0	M_1	M_2	M_3	M_4
M_0	0	1	0	0	0
M_1	0	0	1/2	1/2	0
M_2	0	0	0	0	1
M_3	0	0	0	0	1
M_4	1	0	0	0	0

The solution of this matrix is a vector of steady-state probabilities:

$$\boldsymbol{u} = \begin{bmatrix} 0.250 \\ 0.250 \\ 0.125 \\ 0.125 \\ 0.250 \end{bmatrix}$$

It is then possible to quantify the entropy of the presented example:

$$H(PN) = 3 * (0.250 \log_2 0.250) + 2 * (0.125 \log_2 0.125)) = 2.25$$

For this example the upper bound of uncertainty is $\log_2 5 = 2.3219$. Degree of uncertainty itself can be quantified as a percentage of the calculated entropy to maximum entropy, according to the formula $H(PN)/\log_2 n$, i.e. 2.25/2.3219 = 0.969. This result can be loosely interpreted as the fact that uncertainty of sample Petri net reaches 96.9%, which can be classified as a high degree of uncertainty. And with that pose problems, such as low readability, interpretability, predictability, and other indicators.

6 DISCUSSION

Measurement of uncertainty in process models can be an indicator for reasoning about the explanatory power of these models. Mainly the ability to support different managerial decisions associated with the prediction of the system behaviour under defined probabilities (transition matrix). Degree of uncertainty is usually influenced by a number of elements that contains a process model. These elements include OR, XOR, AND, LOOP. Their arrangement in the process model then implies its uncertainty. Finally, the main influences for the amount of uncertainty in the process model are the probabilities associated witch each branching path (e.g. OR-split). Another approach of uncertainty measurement, which uses quantification of individual substructures in model at different levels of abstraction, is defined in (Jung et al., 2011). That approach measures the structural uncertainty of process models, depending on the location of the above-mentioned components (OR, AND, etc.). Approach defined in this work quantifies uncertainty using concepts of Petri nets with relation to Markov chains. This approach also allows the measurement uncertainty in any model that can be modelled as a Petri net. Thereby is for instance possible to use multiple tokens in the model or implicitly defined multiplicity of edges (arcs).

Advantages of this Approach

- Universal metric for measuring the uncertainty of process models that can be modelled using Petri nets.
- The possibility of using the verification features of Petri nets.
- Clearly defined boundaries of uncertainty (interval $< 0, 1 >$).
- Possibility to set specific probabilities for branching in the model.

Disadvantages of this Approach

- Fundamental deficiencies of Petri nets in general, i.e. state explosion, restrictions based on definitions, etc.

7 CONCLUSIONS AND FUTURE WORK

In this paper was defined method for calculating the uncertainty of any process model, which can be modelled by Petri net. The actual uncertainty quantification is based on the measurement of entropy on the set of all reachable marking of Petri net and its steady-state probabilities. On the prime example is presented the calculation of the uncertainty.

One of the relative weaknesses of this approach is non-implicit definition of branching probabilities, i.e. the need to explicitly define these probabilities in the transition matrix (or not consider probabilities at all). Therefore, the future research will be focused on defining this method using stochastic Petri nets, which implicitly define probability rates of transitions in its definition.

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