Tetrachromatic Metamerism
A Discrete, Mathematical Characterization

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Abstract: Two light beams that are seen as of having the same colour but that have different spectra are said to be metameric. The colour of a light beam is based on the reading of severel photodetectors with different spectral responses and metamerism results when a set of photodetectors is unable to resolve two spectra. The spectra are then said to be metameric. We are interested in exploring the concept of metamerism in the tetrachromatic case. Applications are in computer vision, computational photography and satellite imaginery, for example.

1 INTRODUCTION

Two light spectra are said to be metameric when the corresponding lights look of the same colour. For example, a spectral (i.e. of energy at a unique wavelength) yellow light beam and an appropriate combination of spectral beam lights, green and red. We stress the point that it is pairs of spectra, and not pairs of colors, that are metameric.

We explore the concept of metamerism in the tetrachromatic case, mainly from a mathematical viewpoint. Applications are in computer vision, nonhuman biological vision, computational photography and satellite imaginery, for example.

Given four photodectectors with spectral sensitivity curves $\mathbf{w}(\lambda)$, $\mathbf{x}(\lambda)$, $\mathbf{y}(\lambda)$, and $\mathbf{z}(\lambda)$, and the spectrum $\mathbf{s}(\lambda)$ of a light beam that falls on the surface of each, the corresponding responses are given by $c_w = \int \mathbf{s}(\lambda)\mathbf{w}(\lambda)$, $c_x = \int \mathbf{s}(\lambda)x(\lambda)$, $c_y = \int \mathbf{s}(\lambda)\mathbf{y}(\lambda)$ and $c_z = \int \mathbf{s}(\lambda)\mathbf{z}(\lambda)$. The integrals measure the area below the spectrum curve (i.e. the radiant energy) as "seen through" each of the sensitivity curves. In a sense, the sensitivity curves *aperture sample* the spectrum. In such a tetrachromatic a vision system[1], two spectra giving rise to the same responses c_w, c_x, c_y and c_z will be undistinguishble by the photodetectors and will be said to be metameric. The point $\mathbf{c} = [c_w, c_x, c_y, c_z] \in \mathcal{R}^4$ will be called a *colour point*; here, $\mathcal{R}$ denotes the set of the real numbers.

Thus, in going from $\mathbf{s}(\lambda)$ to $\mathbf{c} = [c_w, c_x, c_y, c_z]$, you take four *aperture samples* of $\mathbf{s}$ and metamerism results when the photodetectors are unable to resolve two spectra. This is unavoidable if you consider that a set of four photoreceptors linearly[2] maps the graph curve of each spectrum function $\mathbf{s} : [\lambda_{min}, \lambda_{max}] \to [0, \infty)$ into a point on the "16-tant[3]" $\mathcal{R}^{4+}$ that we denote also as [+, +, +, +]:= $\{[t_1, t_2, t_3, t_4] : t_i \geq 0\}$ of $\mathcal{R}^4$.

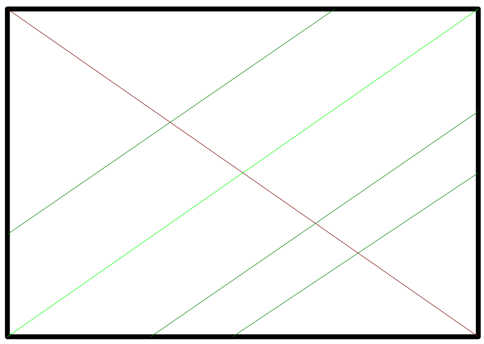

Figure 1: The rectangle is meant to be the domain space of a linear transformation; the point at the center is the origin of the space. The light green line is meant to be the kernel of the transformation, the green lines are cosets and the red line is the orthogonal complement of the kernel.

[1]In our case, we are old-world, fruguivore, trichromatic primates and most of us have exactly three types of photopigment in the main receptor layer of our retinae, but many animals are tetrachromatic.

[2]The fact that the irradiance of the light beam is nonlinearly contracted to a bounded luminance is being overlooked here; nevertheless, regarding the hue, things are pretty much linear..

[3]Analogously to the cuadrants of the plane and the octants of 3-space.

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In order to use the machinery of linear spaces with the transformation $\mathbf{s} \mapsto \mathbf{c}$, we must allow both spectra and color points to take on negative values as well. The resulting sets of spectra $\mathcal{R}^{[\lambda_{min},\lambda_{max}]}$ and of colours $\mathcal{R}^4$ are now linear spaces and will be called sets of *virtual spectra* and of *virtual colours*, respectively. When a virtual spectrum is nonnegative we say that is is *realizable* and, likewise, when the components of a colour are nonnegative, we say that it is a *realizable* colour.

The kernel of the transformation $\mathcal{R}^{[\lambda_{min},\lambda_{max}]} \rightarrow \mathcal{R}^4$ that maps $\mathbf{s} \mapsto \mathbf{c}$, is the set of spectra that are mapped to the colour point [0, 0, 0, 0], called here black. The spectra in each corresponding coset of spectra are mapped to the same (colour) point on $\mathcal{R}^4$; this is the main idea behind a mathematization of the phenomenon of metamerism. See Figure 1.

In our digital, technical world, magnitudes are made discrete; a camera aperture-samples the spectrum of the light at each of many small spatial regions or pixels. In addition to this, we assume that such sampling is done over an already sampled spectrum, sampled at a much more finer scale, e.g. every 10 nm, or so. Thus, the interval set of wavelengths $[\lambda_{min}, \lambda_{max}] \subset \mathcal{R}$ is converted to a finite sequence $\{\lambda_1 = \lambda_{min}, \lambda_2, \lambda_3, \ldots \lambda_N = \lambda_{max}\}$ of wavelengths, and the transformation from spectra to colors is now of the form $\mathcal{R}^N \rightarrow \mathcal{R}^4$, considerably simpler and yet a good approximate model. Integrals become dot products, i.e. $c_w = \mathbf{w}.\mathbf{s} = \sum w_i s_i$, $c_x = \mathbf{x}.\mathbf{s} = \sum x_i s_i$, $c_y = \mathbf{y}.\mathbf{s} = \sum y_i s_i$ and $c_z = \mathbf{z}.\mathbf{s} = \sum z_i s_i$.

Most mammals are dichromatic[4] and it has been argued that the L photopigment evolved in old-world monkeys as it resulted advantageous in the appraisal of the ripeness of fruits when seen from the distance; or, since male dichromacy is common in such primates, that it evolved providing females with health cues regarding potential mates. In biological vision, tetrachromacy is found in fish, birds, reptiles and in many invertebrates; the mantis shrimp is 12-chromatic and sees in the range from 300 nm to 700 nm. In satellite imaginery, the bands may be many but you may restrict to R, G, B plus either NIR or UV. In both cases, the bands include some amount of overlap but are mostly disjoint; unlike the case of a recently developed detector for photography that, in addition to the usual R, G and B pixels of a Bayer, array sensor, it includes unfiltered (other than by the glass of the lens of the camera and the package of the sensor) ”panchromatic” pixels.

Spectral lights are perceived as more saturated than more wide-band lights; thus, the spectral yellow might appear a bit more saturated, than the yellow that results from the mixture of green and red. In what we call *hue metamerism*, luminance differences of the light beams can be considered immaterial, as well as the saturation, up to a degree. Two colors may have the same hue but different luminance and different chromatic saturation; correspondingly, a relaxed type of metamerism may be also exploited in computer vision systems; differences in luminance may be due to differences in illumination intensity and differences in saturation may be due to atmospheric conditions but not to different spectral reflectances of surfaces. In this line, it is useful to consider a hypercube of photodetector responses and identify sets of constant hue or *chromatic triangles*, in it (Restrepo, 2013b), (Restrepo, 2013a).

Figure 2: Satellite RGB image and processed NIR-RGB image. From (Restrepo, 2013b).

[4] Most mammals have cones S (short wavelengths) and M (medium wavelengths) but no cone L (large wavelengths). See (Jevbratt, 2013).

2 TETRACHROMATIC METAMERISM

Let **w**, **x**, **y** and **z** be four linearly independent, N-vectors of samples of the spectral responses, at a set of wavelengths $\lambda_1, ...\lambda_N$, of four photodetectors; also, let the (corresponding samples of the) light spectrum be given by **s**. Denote the "colour" response of the photodetector set by $\mathbf{c} = [c_w, c_x, c_y, c_z]$. Assume then that the response to a light beam falling on four such, nearly placed, photodetectors is given by

$$\mathbf{c}^T = \begin{bmatrix} \mathbf{w} \\ \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{bmatrix} \mathbf{s}^T =: \mathbf{M}\mathbf{s}^T$$

or

$$\begin{bmatrix} c_w \\ c_x \\ c_y \\ c_z \end{bmatrix} = \begin{bmatrix} w_1 & \dots & w_N \\ x_1 & \dots & x_N \\ y_1 & \dots & y_N \\ z_1 & \dots & z_N \end{bmatrix} \begin{bmatrix} s_1 \\ . \\ .. \\ s_N \end{bmatrix}$$

This provides a linear transformation $\mathcal{R}^N \to \mathcal{R}^4$, $\mathbf{s} \mapsto \mathbf{r}$, that reduces the dimensionality from N to 4. **M** has full rank and its entries are nonnegative and, typically, positive. Thus, for a nonnegative **s**, **c** is nonnegative: $\mathbf{c} \in \mathcal{R}^{4+}$. The kernel **K** of this transformation is given by the set of vectors **k** for which

$$\mathbf{Mk} = \begin{bmatrix} \mathbf{w} \\ \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{bmatrix} \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus, **K**, the set of the *metameric blacks*, is the space of vectors orthogonal to (each element of) the subspace $\mathbf{L} := span\{\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}\} = \{\mathbf{aM} : \mathbf{a} \in \mathcal{R}^4\}$, which is isomorphic to $\mathcal{R}^4$. Also, any two spectra ${}_1\mathbf{s}$ and ${}_2\mathbf{s}$ such that ${}_1\mathbf{s} -_2 \mathbf{s} \in \mathbf{L}^\perp = \mathbf{K}$, produce the same colour response $\mathbf{c} = [c_w, c_x, c_y, c_z]$. **L** has dimension 4 and $\mathbf{L}^\perp$ has dimension $N-4$; also, $\mathbf{MM}^T : \mathcal{R}^4 \to \mathcal{R}^4$ is invertible. **K** contains "spectra" (we might call them *virtual spectra*) that are neither nonnegative nor nonpositive[5]. The cosets $\mathbf{s}+\mathbf{K} := \{\mathbf{s}+\mathbf{k} : \mathbf{s} \in \mathcal{R}^N, \mathbf{k} \in \mathbf{K}\}$ provide a partition of $\mathcal{R}^N$. In a decomposition $\mathbf{s} = \mathbf{f}+\mathbf{k}$, $\mathbf{f} \in \mathbf{L}$, $\mathbf{k} \in \mathbf{K}$, which is unique, **f** is called a *fundamental metamer* and **k** is called a *metameric black*. The spectra in the coset $\mathbf{f}+\mathbf{K}$ are said to be metameric and are mapped by **M** to the same colour point $\mathbf{c} \in \mathcal{R}^4$; only the nonnegative spectra in such coset are realizable, the remaining are merely virtual.

[5]The spectra in $\mathcal{R}^N$ that are nonnegative are those in the *wedge* or "2^N-tant" [+, +, ... +]:= $\mathcal{R}^{N+}$

2.1 A Basis for K

Calling the colour point [0, 0, 0, 0] black, then **K** is the set of spectra that "evoke" the colour black; call them metameric blacks. Since the components of **M** are nonnegative, the only nonnegative spectrum that is a metameric black is the 0 spectrum; all other metameric black spectra include both positive and negative components.

Cohen's method (Cohen and Kappauf, 1982), based on CIE data, consists of finding **f** as $\mathbf{f} = [\mathbf{M}^T(\mathbf{MM}^T)^{-1}\mathbf{M}]\mathbf{s}$ and then writting $\mathbf{k} = \mathbf{s}-\mathbf{f}$.

We derive a basis for **K** of narrow-band spectra in a 4-step process where 4 triangular, sparse matrices of row vectors of local support are derived. In the first matrix ${}_1\mathbf{A}$ you have a basis for the orthogonal complement of span$\{\mathbf{w}\}$, in the second one ${}_2\mathbf{A}$, a basis for the orthogonal complement of span$\{\mathbf{w}, \mathbf{x}\}$, then, in ${}_3\mathbf{A}$, a basis for span$\{\mathbf{w}, \mathbf{x}, \mathbf{y}\}^\perp$ and finally, in ${}_4\mathbf{A}$, a basis for span$\{\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}\}^\perp$. We assume that the components of **w**, **x**, **y** and **z**, are positive so that the matrix ${}_1\mathbf{A}$ below is computable and also that each of the matrices ${}_2\mathbf{A}$, ${}_3\mathbf{A}$ and ${}_4\mathbf{A}$, as defined below, are computable.

Let ${}_1\mathbf{A}$ be the $N \times (N-1)$ matrix with i^{th} row of the form $[0, ..., 0, 1, -w_i/w_{i+1}, 0, ..., 0]$; thus, ${}_1M$ has a diagonal of 1's. Clearly, each row of ${}_1\mathbf{A}$ is orthogonal to **w** and, since linearly independent, they provide a basis for span$\{\mathbf{w}\}^\perp$.

Let each row of ${}_2\mathbf{A}$ result from linearly combining each pair of consecutive rows of from ${}_1\mathbf{A}$. In this way, each row is still orthogonal span$\{\mathbf{w}\}^\perp$ and, by using appropriate weights in the combination, you can make it also orthogonal to span$\{\mathbf{x}\}^\perp$. In fact, let the i^{th} row of the $N \times (N-2)$ matrix ${}_2\mathbf{A}$ be given by

$[0, ..., 0, 1, m_{i,i+1} + \beta_i m_{i+1,i+1}, \beta_i m_{i+1,1+2}, 0, ..., 0]$

where the m's are the components of ${}_1\mathbf{A}$, and

$\beta_i = -\frac{x_i + m_{i,i+1}x_{i+1}}{m_{i+1,i+1}x_{i+1} + m_{i+1,i+2}x_{i+2}}$;

again, the diagonal of ${}_2\mathbf{A}$ is a diagonal of 1's. Likewise, by making sure a certain linear combination of each two consecutive rows in ${}_2\mathbf{A}$ is orthogonal to **y**, you get the $N \times (N-3)$-matrix ${}_3\mathbf{A}$ with i^{th} row of the form

$[0, ..., 0, 1, m_{i,i+1} + \beta m_{i+1,i+1}, m_{i,i+2}$
$+\beta m_{i+1,i+2}, \beta m_{i+1,i+3}, 0, ..., 0]$

where the m's are now the components of ${}_2\mathbf{A}$ and

$\beta_i = -\frac{y_i + y_{i+1}m_{i,i+1} + y_{i+2}m_{i,i+2}}{y_{i+1}m_{i+1,i+1} + y_{i+2}m_{i+1,i+2} + y_{i+3}m_{i+1,i+3}}$.

The diagonal of ${}_3\mathbf{A}$ is a diagonal of ones. Finally, a linear combination of each two consecutive rows in ${}_3\mathbf{A}$ that is orthogonal to **z**, provides the $N \times (N-4)$-matrix ${}_4\mathbf{A}$ with i^{th} row

$[0, ..., 0, 1, m_{i,i+1} + \beta m_{i+1,i+1}, m_{i,i+2}$
$+\beta m_{i+1,i+2}, m_{i,i+3} + \beta m_{i+1,i+3}, \beta m_{i+1,1+4}, 0,$
$..., 0]$,

where the m's are now the components of M^3 and

$\beta_i = -\frac{z_i + z_{i+1} m_{i,i+1} + z_{i+2} m_{i,i+2} + z_{i+3} m_{i,i+3}}{z_{i+1} m_{i+1,i+1} + z_{i+2} m_{i+1,i+2} + z_{i+3} m_{i+1,i+3} + z_{i+4} m_{i+1,i+4}}$. See Fig. 6. Let the rows of the $N \times (N-4)$-matrix $\mathbf{B} :=_4 \mathbf{A}$ denote this resulting base for the kernel $\mathbf{K}$ of $\mathbf{M}$.

Note that the rows in each of these matrices are linearly independent due to their localized support. The rows in $_1\mathbf{A}$ have support of length 2, those in $_2\mathbf{A}$ have support of length 3, those in $_3\mathbf{A}$ have support of length 4 and those in the basis $\mathbf{B}$ of $\mathbf{L}^\perp = \mathbf{K}$ have support of length 5. The existence of β in each case is not so surprising due to this localization; that is, for example, it is not too difficult to find constants α_1 and β_1 so that $(\alpha[1, -w_1/w_2, 0] + \beta[0, 1, -w_2/w_3]).[x_1, x_2, x_3] = [\alpha, -\alpha w_1/w_2 + \beta, -\beta w_2/w_3].[x_1, x_2, x_3] = \alpha x_1 + (-\alpha w_1/w_2 + \beta)x_2 - (\beta w_2/w_3)x_3 = 0$; letting $\alpha = 1$, you only need $x_1 - w_1 x_2/w_2 + \beta(x_2 - w_2 x_3/w_3) = 0$, i. e., $x_2 \neq \frac{w_2}{w_3} x_3$ or $\frac{x_2}{x_3} \neq \frac{w_2}{w_3}$. In fact, when computing $_2\mathbf{A}$, each β_i can be also expressed as $\beta_i = -\frac{x_i - \frac{w_i}{w_{i+1}} x_{i+1}}{x_{i+1} - \frac{w_{i+1}}{w_{i+2}} x_{i+2}}$. Similar formulas of β can be derived for the remaining cases of $_3\mathbf{A}$ and $_4\mathbf{A}$

The addition of a scaled element of the so-obtained basis for $\mathbf{K}$ is the addition of a metameric black that alters the spectrum in a very narrow region of it producing a new metamer. Large peaks in a base element of $\mathbf{K}$ may indicate indicate insensitivity to certain wavelengths.

2.2 Sets of Metameric Spectra

The set of spectra $\mathcal{R}^N$ is partitioned into cosets of the form $\mathbf{K} + \mathbf{s}$, $\mathbf{s} \in \mathcal{R}^N$. To each colour point $\mathbf{c} = [w, x, y, z]^T$ in the hypercube, there corresponds the coset $\mathbf{S_c}$ of dimension N-4, of spectra (not necessarily realizable as physical spectra) that are mapped by the matrix

$$\mathbf{M} = \begin{bmatrix} \mathbf{w} \\ \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{bmatrix}$$

to such colour point. To find $\mathbf{S_c}$, you find a spectrum vector $\mathbf{s}$ for which $\mathbf{Ms} = \mathbf{c}$ and then write $\mathbf{S_c} = \mathbf{K} + \mathbf{s}$. To find one such $\mathbf{s}$, choose a 4x4 matrix $\mathbf{N}$ given by four columns of $\mathbf{M}$, say the $i^{th}, j^{th}, k^{th}, l^{th}$ columns:

$$\mathbf{N} = \begin{bmatrix} w_i & w_j & w_k & w_l \\ x_i & x_j & x_k & x_l \\ y_i & y_j & y_k & y_l \\ z_i & z_j & z_k & z_l \end{bmatrix}$$

We assume that $\mathbf{N}$ is invertible; in fact we choose one such matrix $\mathbf{N}$ having highest absolute determinant so that the computation of its inverse is more accurate. Regarding the possible values of the absolute value of the determinant, there are $\binom{76}{4} = 1.282.975$ choices of i, j, k, l to consider. Once one such matrix is chosen, put $\mathbf{t} = \mathbf{N}^{-1}\mathbf{c}$ and

$$\mathbf{s} = [0, \ldots t_i, 0 \ldots t_j, 0, \ldots t_k, 0, \ldots t_l, 0, \ldots]^T$$

$\mathbf{s}$ is not necessarily in $\mathbf{L}$, i.e. it is not necessarily a fundamental metamer; also, $\mathbf{t}$ may have negative components; in such case, a spectrum $\mathbf{s}$ that is nonzero only at positions i, j, k, l and produces colour $\mathbf{c}$, is not physically realizable. Only the nonnegative spectral photoreceptor vectors in $\mathbf{K} + \mathbf{s}$ are of realizable. It is possible that a realizable colour not be the image of a realizable spectrum.

If you are designing a tetrachromatic imaging system and do not want spectra $_1\mathbf{s}$ and $_2\mathbf{s}$ to be metameric, at least one of the pairs $\mathbf{w}.{_1\mathbf{s}}$ and $\mathbf{w}.{_2\mathbf{s}}$, or $\mathbf{x}.{_1\mathbf{s}}$ and $\mathbf{x}.{_2\mathbf{s}}$, or $\mathbf{y}.{_1\mathbf{s}}$ and $\mathbf{y}.{_2\mathbf{s}}$, or $\mathbf{z}.{_1\mathbf{s}}$ and $\mathbf{z}.{_2\mathbf{s}}$ should be different, in particular, $_1\mathbf{s} -_2 \mathbf{s}$ should be in $\mathbf{L}$ and must not be in $\mathbf{K}$; i.e. $_1\mathbf{s} -_2 \mathbf{s} \in \mathbf{L} - \mathbf{K}$.

3 A CASE EXAMPLE: RGB+PANCHROMATIC

Besides satellites, a source of tetrachromatic images is computational photography. TrueSense Imaging inc. markets a digital image sensor that, in addition to R, G, and B pixels of a Bayer pattern, it includes as well panchromatic pixels[6] in a pattern as shown in Fig. 3. The proportions are 1/4 of green pixels, 1/8 of red pixels, 1/8 of blue pixels and 1/2 panchromatic pixels. Even though the photosensitive transducers respond well into the UV, the microlens blocks wavelengths below 350 nm. The sensor responds in the infrared but the response is negligible above 1050 nm.

The pattern of the color filter array is

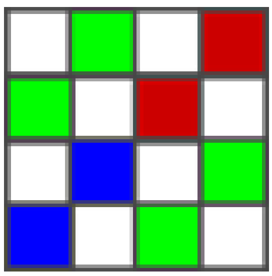

Figure 3: Pattern in the array of the sensor Truesense Imaging KAI-01150: P B P G; B P G P; P G P R; G P R P.

[6] Pixels that are covered by the microlens but that otherwise do not receive filtered light.

For our purposes, we do not need interpolate the data in the pattern array that give rise to the image shown at the top in Figure 4; instead, we downsample each 4×4 pixel block to a tetrachromatic pixel, by averaging the pixels in each band in the block. Thus, even though the original image is 1152×2044, the image we work with is only 287×510 pixels. Also, the bands we use are $w = P$, $x = R$, $y = G$ and $z = B$.

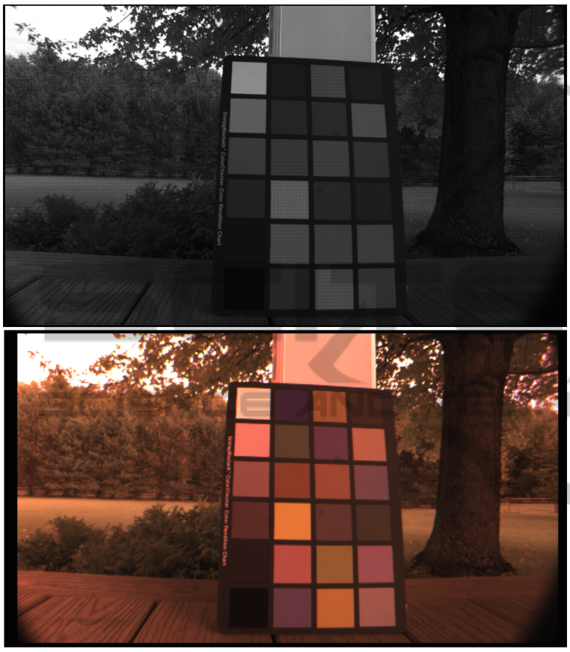

Figure 4: Outdoors, 16-bit, RGBP image of a Macbeth chart; courtesy of Amy Enge. Below, RGB visualization without corrections.

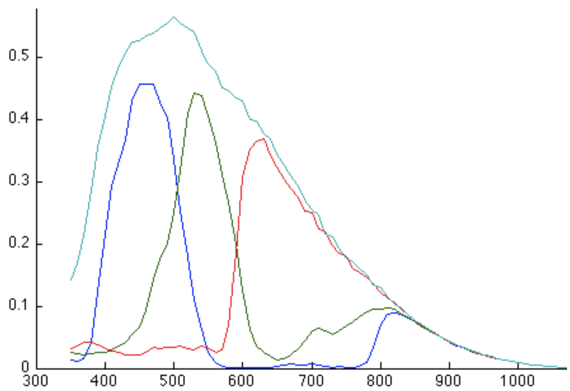

Figure 5: Quantum efficiencies corresponding to Truesense sensors P, R, G and B.

The data provided by TrueSense of the quantum efficiency of each sensor type, at each 10 nm from 350 to 1100 nm, provides 76 data per band. The basis elements in are shown in Figures 6, at the bottom.

The submatrix with largest determinant is given by

$$\mathbf{N} = \begin{bmatrix} p_5 & p_{13} & p_{20} & p_{29} \\ r_5 & r_{13} & r_{20} & r_{29} \\ g_5 & g_{13} & g_{20} & g_{29} \\ b_5 & b_{13} & b_{20} & b_{29} \end{bmatrix}$$

$$= \begin{bmatrix} 0.3544 & 0.5390 & 0.5110 & 0.3769 \\ 0.0378 & 0.0337 & 0.0354 & 0.3688 \\ 0.0261 & 0.1453 & 0.4382 & 0.0262 \\ 0.1233 & 0.4566 & 0.0628 & 0.0013 \end{bmatrix}$$

and has determinant 0.0136 and inverse given by

$$\mathbf{N}^{-1} = \begin{bmatrix} -4.8879 & -5.0124 & -4.1153 & 5.1435 \\ 1.3590 & 1.0476 & 3.4252 & -1.4138 \\ -0.3459 & 2.2207 & -0.9022 & 0.1873 \\ 3.1216 & 0.2047 & 0.1949 & -0.4158 \end{bmatrix}$$

For example, corresponding to colour $\mathbf{c} = [0.25, 0.25, 0.25, 0.75]^T$ you get $\mathbf{t} = [0.3538, 0.3976, 0.3837, 0.5685]^T$ and $\mathbf{s} = [0 \ldots 0.3538, 0 \ldots 0.3976, 0 \ldots 0.3837, 0 \ldots 0.5685, 0 \ldots]$, with nonzero values at coordinates 5, 13, 20 and 29.

3.1 Program Code

In the MATLAB code below, vectors R, G, B, P are the quantum efficiencies. The base for the orthogonal complement is in AL4. Note: here, the matrix **M** used is $R = [\mathbf{r}; \mathbf{g}; \mathbf{b}; \mathbf{p}]^T$

```
AL1= zeros(76,75);
for ii=1:75
   AL1(ii,ii)= 1;
   AL1(ii,ii+1)= -R(ii)/R(ii+1);
end
figure; plot(L,AL1)

AL2= zeros(76,74);
for ii=1:74
   AL2(ii,ii)= 1;
   BETA= -(G(ii) + G(ii+1)*AL1(ii,ii+1)/...
   AL1(ii,ii))/(AL1(ii+1,ii+1)*G(ii+1)+ ...
   AL1(ii+1,ii+2)*G(ii+2));
   AL2(ii,ii+1)= AL1(ii,ii+1)/AL1(ii,ii)+ ...
   BETA*AL1(ii+1,ii+1);
   AL2(ii,ii+2)= BETA*AL1(ii+1,ii+2);
end
figure; plot(L,AL2)

AL3= zeros(76,73);
for ii=1:73
   AL3(ii,ii)= 1;
   BETA= -(B(ii) + B(ii+1)*AL2(ii,ii+1)/...
   AL2(ii,ii) + B(ii+2)*AL2(ii,ii+2)/...
   AL2(ii,ii))/ (B(ii+1)*AL2(ii+1,ii+1) ...
   + B(ii+2)*AL2(ii+1,ii+2) + B(ii+3)*...
   AL2(ii+1,ii+3));
   AL3(ii,ii+1)= AL2(ii,ii+1)/AL2(ii,ii)+ ...
```

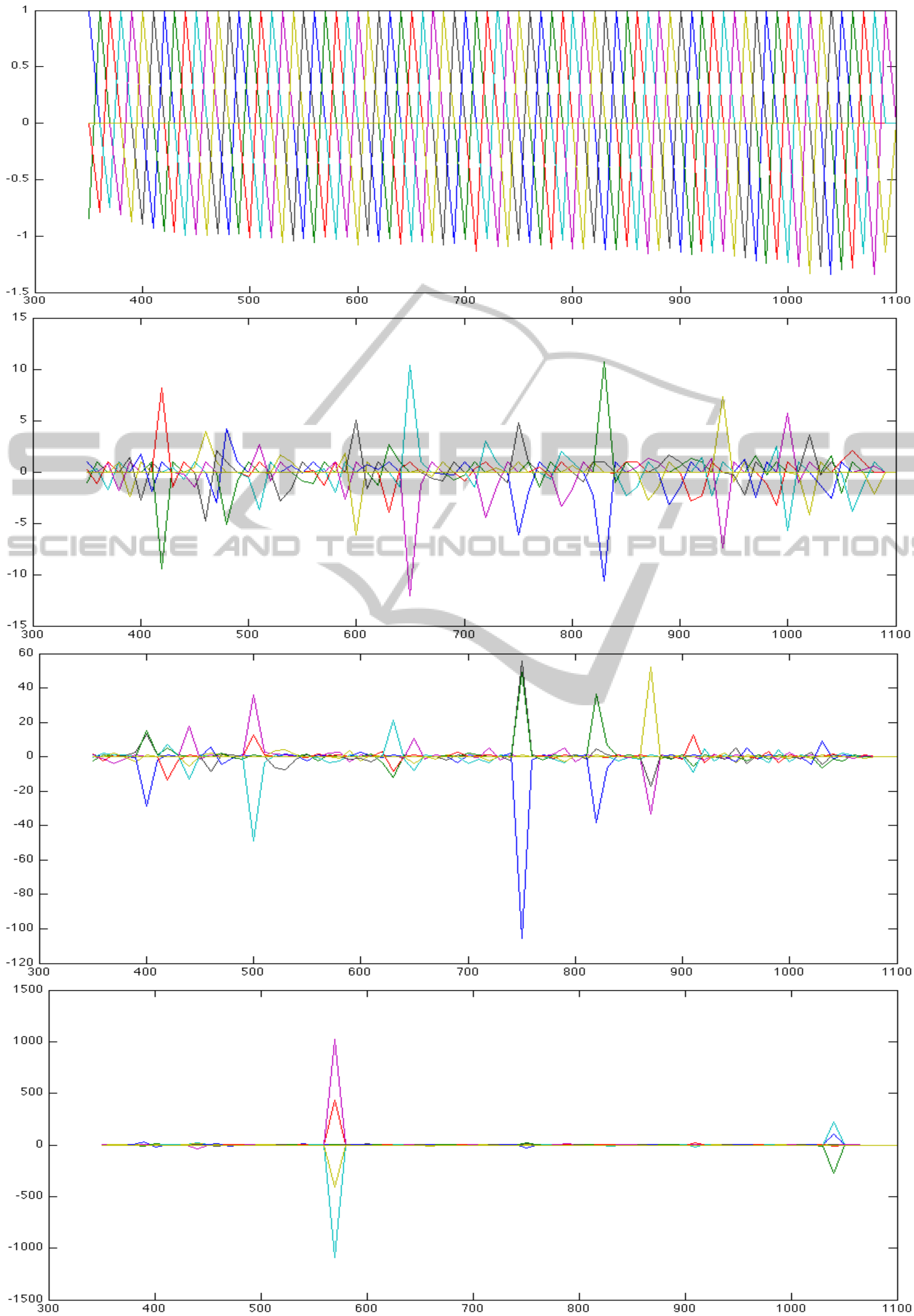

Figure 6: Obtention of the kernel $\mathbf{K}$ (in bottom row) of $[\mathbf{p},\mathbf{r},\mathbf{g},\mathbf{b}]^T : \mathbf{R}^{76} \to \mathbf{R}^4$. In each graph, respectively from above, 75, 74, 73 and 72 row vectors are plotted.

```
   BETA*AL2(ii+1,ii+1);
   AL3(ii,ii+2)= AL2(ii,ii+2)/AL2(ii,ii)+ ...
   BETA*AL2(ii+1,ii+2);
   AL3(ii,ii+3)= BETA*AL2(ii+1,ii+3);
end
figure; plot(L,AL3)
```

```
\end{small}
AL4= zeros(76,72);
for ii=1:72
   AL4(ii,ii)= 1;
   BETA= -(P(ii) + P(ii+1)*AL3(ii,ii+1)/...
   AL3(ii,ii) + P(ii+2)*AL3(ii,ii+2)/AL3(ii,ii)...
   + P(ii+3)*AL3(ii,ii+3)/AL3(ii,ii))/(P(ii+1)*...
   AL3(ii+1,ii+1) + P(ii+2)*AL3(ii+1,ii+2) +...
    P(ii+3)*AL3(ii+1,ii+3) + P(ii+4)*...
    AL3(ii+1,ii+4));
   AL4(ii,ii+1)= AL3(ii,ii+1)/AL3(ii,ii) + ...
   BETA*AL3(ii+1,ii+1);
   AL4(ii,ii+2)= AL3(ii,ii+2)/AL3(ii,ii) + ...
   BETA*AL3(ii+1,ii+2);
   AL4(ii,ii+3)= AL3(ii,ii+3)/AL3(ii,ii) + ...
   BETA*AL3(ii+1,ii+3);
   AL4(ii,ii+4)= BETA*AL3(ii+1,ii+4);
end
```

In the MATLAB code below, get 4x4 submatrix with largest determinant, then get **t** (METAMER) for, e.g. $\mathbf{c} = [\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}]$.

```
%
AL= transpose(AL);
iii=1
DETMAX= 0;
for ii = 1:73
  for jj= ii+1:74
      for kk = jj+1:75
        for ll = kk+1:76
AL4x4=[AL(1, ii), AL(1,jj), AL(1,kk), AL(1, ll);...
      AL(2, ii), AL(2,jj), AL(2,kk), AL(2, ll);...
      AL(3, ii), AL(3,jj), AL(3,kk), AL(3, ll);...
      AL(4, ii), AL(4,jj), AL(4,kk), AL(4, ll);];
DET= abs(det(AL4x4));
iii=iii+1;
if DET > DETMAX;
    DETMAX= DET;
    ii1=ii;
    jj1=jj;
    kk1=kk;
    ll1=ll;
    AL4= AL4x4;
end
    end
    end
    end
end

iii= iii-1
AL4
det(AL4)
AL4INV=inv(AL4)

METAMER= AL4INV*[0.25;0.25;0.25;0.25]
```

4 TETRACHROMATIC HUE METAMERISM

For us humans, two colours may have the same saturation, the same luminance or the same hue. When studying the colour vision of a tetrachromatic animal, it may be interesting to design an experiment to find out if the animal can distinguish hue while disregarding luminance and saturation. In this sense, we call two spectra hue-metameric if they give rise to colour points on the same *chromatic triangle* (Restrepo, 2011); see Figure 7. In the colour hypercube (Restrepo, 2012) you also have the achromatic segment and instead of a chromatic hexagon, you have a chromatic icositetrahedron; the triangles having as base the achromatic segment and as opposing a vertex a point in the chromatic icositetrahedron are called chromatic triangles and all colours in each such triangle are said to have the same hue.

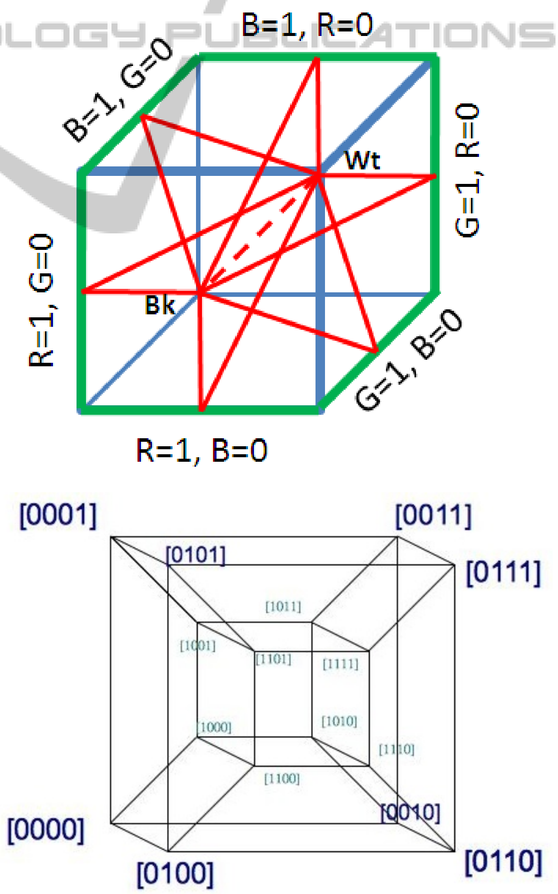

Figure 7: Chromatic triangles in RGB cube and hypercube.

5 CONCLUSIONS

Tetrachromacy and in particular tetrachromatic metamerism is a subject well worth of attention. It has applications both in computer vision and in the modeling of biological vision systems. The subject of tetrachromatic metamerism in computer vision has applications for example in detection, in satellite images.

In our case, we have three types of cone photopigment $S(\lambda)$, $M(\lambda)$ and $S(\lambda)$ and it may be argued that from them, three other channels $L+M+S$, $(L+S)-M$ and $(L+M)-S$ are derived. It is an interesting fact of our colour vision that we have four perceptual unique hues red, green, blue and yellow; perhaps they are in a one-to-one correspondence with the three channels $S+L$ ("band-stop"), M (band-pass), S (low-pass) and $L+M$ (high-pass). The fact that the L channel does not appear in an isolated form here, might have to do with the fact that it was the last to evolve.

It would be interesting to know how these facts extrapolate in cases of the vision systems tetrachromatic animals. One possibility is that they might perceive 6 unique hues, corresponding to the cases $W+X$ (low-pass), $Y+Z$ (high-pass), $X+Y$ (band-pass), $W+Z$ (stop-band) and, $W+Y$ and $X+Z$ (alternate band).

In a trichromatic context, (Cohen, 1964) has shown how the reflectance spectra (samped at N=40 wavelengths) of a set of 150 Munsell chips, turned out to be nearly three-dimensional; i.e. each spectrum is nearly a linear combination of certain three spectra. The analysis of large sets of natural reflectance spectra surely gives interesting results.

In a tetrachromatic vision system, the use detector with a bell response curve having a peak between those of the S and M detectors, could prove to be useful to differentiate between certain types of cyan allowing the perception a certain type of cyan as a unique and not as a combination of green and blue. This would be certainly useful for marine vision since short-wavelength light penetrates water more than other wavelengths.

Typically, receptor curves are unimodal. In biology, although not always in engineering as the example in Section 3 shows, each receptor curve "aperure-samples" the visible spectrum, each sampling the energy in a, maybe overlapping, interval. In engineering, the use of detectors of comb spectra might be useful as well.

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