Belief Revision on Modal Accessibility Relations

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Abstract: In order to model the changing beliefs of an agent, one must actually address two distinct issues. First, one must devise a model of static beliefs that accurately captures the appropriate notions of incompleteness and uncertainty. Second, one must define appropriate operations to model the way beliefs are modified in response to different events. Historically, the former is addressed through the use of modal logics and the latter is addressed through belief change operators. However, these two formal approaches are not particularly complementary; the normal representation of belief in a modal logic is not suitable for revision using standard belief change operators. In this paper, we introduce a new modal logic that uses the accessibility relation to encode epistemic entrenchment, and we demonstrate that this logic captures AGM revision. We consider the suitability of our new representation of belief, and we discuss potential advantages to be exploited in future work.

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1 INTRODUCTION

The study of *belief revision* is concerned with the manner in which an agent's beliefs change in response to new information. The formalization of dynamic beliefs has been studied extensively by modal logicians and by AI researchers. In modal logic, there is an established representation of beliefs in terms of accessibility relations in Kripke structures(van Ditmarsch et al., 2007). The standard approach to modeling dyanmic beliefs is then through transformations on structures that induce new beliefs. In this paper, we argue that this representation of (static) beliefs is not appropriate, if we are interested in capturing the well known AGM approach to belief revision in a Kripke structure. We propose an alternative use of accessibility relations that encodes the relative plausibility of different states, thereby allowing us to explictly capture AGM revision in a modal setting.

Broadly speaking, work based on modal logic has tended to address dynamic beliefs through what could be called a bottom-up approach. In this approach, we start with an expressive representation of *static beliefs*, and then work towards the development of suitable change operations. Modal approaches to belief change often focus on difficult problems related to multi-agent belief revision, or the logic of public announcements. In the AI community, research on belief revision has taken a top-down approach: it is common to represent the beliefs of an agent as a simple set of propositional formulas, but then the belief change operators need to incorporate some notion of plausibility or entrenchment.

The AGM approach to belief revision (Alchourrón et al., 1985) is a representative example of work in the top-down approach, and there is no universally accepted modal formulation of AGM revision. We would like to formulate such a logic without introducing any new machinery in terms of ranking functions or transformations on Kripke structures. In other words, we would like to use the Kripke structure itself to encode the entrenchment of beliefs. By doing so, we are likely to lose the simple modal intuition of accessible alternative worlds. We argue, however, that such a notion is not appropriate for AGM revision in any event, as it allows too much explict information about the structure of beliefs with respect to a change in the world. The end result is a new model conception of belief that provides an exact characterization of AGM revision. The utility of this model for practical application is left for future work, as this is primarily an expository paper focused on introducing a new modal approach to belief revision.

The main contribution of this position paper is the introduction of a simple modal characterization of AGM revision. This characterization has not been specified previously because it is somewhat unnatural from the perspective of modal logic. However, AGM revision itself is somewhat unnatural from the perspective of modal logic; so this is not surprising. The

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second contribution of this paper is a kind of completeness result, demonstrating that every suitable binary Kripke structure defines an AGM revision operator. However, we need to make substantive assumptions about the philosophical significance of "accessibility." This work describes work in progress, returning to the logical roots of AGM revision to solidfy the foundations of the approach with respect to the formal structures of modal logic.

2 BACKGROUND

2.1 Doxastic Logic

We assume the reader is familiar with modal logic, as introduced in (Chellas, 1980). Briefly, modal logic is an extension of propositional logic where $\Box\phi$ is a sentence whenever ϕ is a sentence. The semantics of modal logic is defined in terms of Kripke structures. Let P be a propositional vocabulary. A Kripke structure is a triple $\mathcal{M} = \langle W, R, v\rangle$ where W is a set of possible worlds, R is a binary relation on W, and v is a valuation function that maps every world to an interpretation of P. In modal logic, the entailment relation $\models$ is defined with respect to a pair consisting of a Kripke structure and a possible world.

Different modal logics can be defined axiomatically, or by restricting the accessibility relation R. For our purposes, two modal logics will be important. First, we will be interested in $KD45$, which is the standard doxastic logic. The logic $KD45$ is characterized by allowing only Kripke structures where R has the following properties:

1. R is serial: For all $w \in W$ there exists $x \in W$ such that Rwx.
2. R is transitive: For all $w,x,y \in W$, if Rwx and Rxy then Rwy.
3. R is euclidean: For all $w,x,y \in W$, if Rwx and Rxz then Ryz.

For any $w \in W$, let $R_w = \{x \mid Rwx\}$. Accessibility relations in $KD45$ have the property that R is universal on the set R_w, for each w. It is common to use the symbol B for the modal operator in $KD45$, and interpret $\mathcal{M}, w \models B\phi$ to mean "in world w of the structure $\mathcal{M}$, it is believed that ϕ is true." In order to model the beliefs of n agents, we introduce n modal operators B_i each with a corresponding accessibility relation R_i. This gives the modal logic $KD45_n$.

The second modal logic that will be important in this paper is the modal logic KD, which is the logic obtained by allowing only serial accessibility relations. Hence, every $KD45$ structure is a KD structure but the converse is not true. The logic KD_n is defined in the obvious manner.

2.2 AGM Belief Revision

In the theory of belief revision, the focus is on the *dynamics* of belief as opposed to the *representation* of belief. The basic scenario in belief revision involves a single agent with some a priori beliefs along with some piece of "new" information about the world. The intuition is that the new information is more reliable than the old information, and it must therefore be incorporated.

One of the most influential approaches to belief revision is the AGM approach (Alchourrón et al., 1985). In this approach, the beliefs of an agent are represent by a deductively closed set of formulas called a *belief set*. If we assume a finite signature, we can equivalently represent the beliefs of an agent by a single propositional formula ϕ. An *AGM revision operator* is a binary function $*$ that satisfies the *AGM postulates*, which were reformulated as follows by Katsuno and Mendelzon (Katsuno and Mendelzon, 1992).

[R1] $\phi * \gamma$ implies γ.
[R2] If $\phi \wedge \gamma$ is satisfiable, then $\phi * \gamma \equiv \phi \wedge \gamma$.
[R3] If γ is satisfiable, then $\phi * \gamma$ is satisfiable.
[R4] If $\phi_1 \equiv \phi_2$ and $\gamma_1 \equiv \gamma_2$, then $\phi_1 * \gamma_1 \equiv \phi_2 * \gamma_2$.
[R5] $(\phi * \gamma) \wedge \beta$ implies $\phi * (\gamma \wedge \beta)$.
[R6] If $(\phi * \gamma) \wedge \beta$ is satisfiable, then $\phi * (\gamma \wedge \beta)$ implies $(\phi * \gamma) \wedge \beta$.

Let f be a function that maps every propositional formula ϕ to a total pre-order $\preceq_\phi$ over interpretations. We say that f is a *faithful assignment* if and only if

1. If $s_1, s_2 \models \phi$, then $s_1 =_\phi s_2$.
2. If $s_1 \models \phi$ and $s_2 \not\models \phi$, then $s_1 \prec_\phi s_2$,
3. If $\phi_1 \equiv \phi_2$, then $\preceq_{\phi_1} = \preceq_{\phi_2}$.

Every AGM revision operator can be captured by minimization over a faithful assignment.

Proposition 1. *(Katsuno and Mendelzon, 1992) A revision operator $*$ satisfies [R1]-[R6] just in case there is a faithful assignment that maps each ϕ to an ordering $\preceq_\phi$ such that*

$$s \models \phi * \gamma \iff s \text{ is a } \preceq_\phi -\text{minimal model of } \gamma.$$

3 A MODAL LOGIC OF BELIEF REVISION

3.1 Motivation

The main intuition underlying our approach is the idea that an accessibility relation in a modal logic can be used to define a total pre-order over worlds. This intuition differs from the standard intuition of the logic $KD45$, where the accessibility relation unambiguously associates a *set of believed worlds* with each possible world. We suggest that this is not what is actually required for AGM revision. In AGM revision, the beliefs of an agent correspond to a *set of believed interpretations* (as opposed to believed worlds). The main advantage of $KD45$ is that it permits agents to reason counterfactually in a consistent manner. However, this kind of reasoning is neither possible nor desirable in an AGM setting. The only structure on worlds in AGM revision is a pre-order where the believed worlds are all minimal. However, each believed world leads to a *different* set of plausible alternatives. Agents can not reason counterfactually among the interpretations initially believed possible. As such, we suggest that $KD45$ does not provide a natural logical foundation for AGM belief revision.

In the remainder of this section, we would like to consider whether we need any restrictions on R in order to define belief sets in terms of the set $R_w = \{x \mid Rwx\}$. First, we suggest that R does need to be serial. If R is not serial, then it is possible that no worlds are accessible. In this case, the corresponding belief set is inconsistent. This is not permitted in AGM revision. The rationale for both transitivity and euclideanness depends on allowing an agent to perform some kind of counterfactual reasoning. This is not possible in AGM revision: a belief set is just a set of formulas.

Our approach differs from existing work, such as (Herzig et al., 2004), where the emphasis is on transformations on Kripke structures. A modal logic for AGM revision need only specify the belief set that results from a revision; it need not specify any structure on possible worlds following the revision. Indeed, this is a major weakness of the AGM approach which has lead to a great deal of work on iterated revision.

3.2 A Binary Modal Approach

We define a modal logic that has a single, serial accessibility relation (for each agent). As such, we view the logic as a variation on the modal logic KD. However, instead of the standard unary modal operator, we define a binary modal operator $\Box(\phi,\psi)$. Informally $\mathcal{M},w \models \Box(\phi,\psi)$ will be interpreted to mean that ψ is believed after revising R_w by ϕ. Rather than writing $\Box(\phi,\psi)$, we will adopt the notation $\Box_\phi\psi$ in order to maintain a close connection with standard KD syntax.

We define the modal logic KD_{AGM} for a fixed propositional signature P. Formulas are defined as follows:

1. p is a formula for $p \in P$
2. If ϕ is a formula, then $\neg\phi$ is a formula.
3. If ϕ and ψ are formulas, then $\phi \wedge \psi$ and $\Box_\phi\psi$ are both formulas.

For a Kripke structure $\mathcal{M}$ and a world w, the entailment relation $\models$ is defined in the usual way for atomic formulas, negations and conjunctions. To define the semantics of modal formulas, we need some definitions.

Definition 1. *For any w and v, define $d(w,v)$ to be the minimal number n such that there exists a sequence $w_1,\ldots,w_n$ satisfying these conditions:*

1. *$w_1 = w$ and $w_n = v$*
2. *Rw_iw_{i+1} for $1 \leq i \leq n$.*

If no such path exists, then we say $d(w,v) = \infty$.

We say that $d(w,v)$ is the *distance* between w and v.

Definition 2. *For $w \in W$ and a formula ψ, define $D(w,\phi)$ to be the set of all $v \in W$ satisfying:*

1. *$\mathcal{M},v \models \phi$*
2. *$d(w,v)$ is minimal among worlds satisfying 1.*

Therefore $D(w,\phi)$ denotes the set of worlds satisfying ϕ that are minimally distant from w.

For modal formulas, we define

$$\mathcal{M},w \models \Box_\phi\psi \iff \mathcal{M},v \models \psi \text{ for all } v \in D(w,\phi).$$

Note that this definition is a straightforward extension of the normal definition of $\Box\psi$, except that the quantification is now relativized to the set $D(w,\phi)$. We write $\Box\psi$ as a shorthand notation for $\Box\top\psi$. This "induced" unary modality is clearly a KD modal operator.

3.3 Relationship with AGM Revision

We associate a revision operator with every Kripke structure, according to the following definition.

Definition 3. *For any structure $\mathcal{M}$, define $\phi *_\mathcal{M} \gamma = \psi$ where ψ is a formula such that: $\mathcal{M},w \models B(\gamma,\psi)$ if and only if $w \models \phi$.*

We remark that $*_\mathcal{M}$ is not well-defined if the vocabulary in infinite; in such a case, there may not be a single formula ψ satisfying the definition. The problem is that, if the vocabulary is infinite, then we can

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not be assured that every set of interpretations is defined by a unique formula. However, if we restrict attention to finite vocabularies, this is not a problem.

Proposition 2. *If the underlying vocabulary is finite, then $*_{\mathcal{M}}$ is a well-defined function on formulas.*

We call the operator $*_{\mathcal{M}}$ a partial revision operator because it does not define revision for *every* possible initial belief set. Instead it only specifies the outcome of belief revision for the collection of belief sets defined by the structure $\mathcal{M}$.

Definition 4. $\mathcal{M} = \langle W, R, v \rangle$ *is a* complete structure *if, for each* $\alpha \subseteq 2^P$*, there is exactly one* $w \in W$ *such that* $R_w = \alpha$.

The following result says that complete structures define AGM revision operators.

Proposition 3. *Let $\mathcal{M}$ be a complete KD_{AGM} structure. Then $*_{\mathcal{M}}$ is an AGM revision operator.*

The converse is also true.

Proposition 4. *Let $*$ be an AGM revision operator. Then $* = *_{\mathcal{M}}$ for some complete KD_{AGM} structure $\mathcal{M}$.*

Hence, KD_{AMG} structures provide an equivalent characterization of AGM revision. Syntactically, we would define the modal logic KD_{AGM} in terms of a set of axioms. We leave this problem for future work.

4 DISCUSSION

There has been a large body of research on the relation between belief revision and modal epistemic logic. Of particular note is the work on dynamic epistemic logic, originating with (Baltag et al., 1998) and (Baltag and Moss, 2004). However, most of the work in this tradition is focused on providing some kind of transformation on Kripke structures. The idea is to represent the initial beliefs of agents in a Kripke structure, and then provide a systematic way to define a new Kripke structure that represents the beliefs after some event occurs.

Relating work in dynamic epistemic logic to AGM revision has proven to be a challenge. In this paper, we propose that the reason this is a challenge is simply because the representation of belief in a Kripke structure is not fundamentally equivalent to the representation in AGM revision. In AGM revision, beliefs are represented by a set of formulas with no additional structure. While a Kripke structure associates a set of formulas with a belief state, there is actually additional structure in the form of a binary accessibility relation that permits counterfactual reasoning. Hence, we argue that a typical Kripke structure actually can express some forms of reasoning about belief that can not be represented in the AGM framework. At the same time, the AGM model of belief revision implicitly requires an ordering over possible interpretations in order to carry out revision. Such an ordering is not immediately available in a the standard KD45 modal approach. So the typical Kripke structure approach does not capture all aspects of the AGM approach.

In this work, we suggest that Kripke structures can in fact be used to reason about belief change in an AGM-like setting; however, we need to take a different perspective on accessibility. If we use the accessibility relation to encode some notion of plausibility, then we have shown that a simple class of Kripke structures can completely characterize the AGM approach to belief revision. However, in order for this to be possible, we lose the Kripke-style intuition about accessibility and "possible worlds."

In future work, we intend to consider formal embeddings of this logic in standard doxastic logic. More importantly, we also intend to explore the application of this logic for reasoning about belief change in multi-agent systems. One of the real disadvantages of the AGM approach is that it provides no capability for one agent to reason about the beliefs of another agent. This has been studied extensively in the modal approach, particularly with respect to public announcements. It is our hope that this new modal formulation of AGM can take advantage of higher level beliefs, while maintaining an explicit notion of entrenchment.

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