

# Faster Approximations of Shortest Geodesic Paths on Polyhedra Through Adaptive Priority Queue

William Robson Schwartz<sup>1</sup>, Pedro Jussieu de Rezende<sup>2</sup> and Helio Pedrini<sup>2</sup>

<sup>1</sup>Department of Computer Science, Universidade Federal de Minas Gerais, Belo Horizonte-MG 31270-010, Brazil

<sup>2</sup>Institute of Computing, University of Campinas, Campinas-SP 13083-852, Brazil

Keywords: Geodesic Paths, Shortest Geodesic Distance, Shortest Path, Dijkstra's Algorithm, Bucketing Data Structure.

Abstract: Computing shortest geodesic paths is a crucial problem in several application areas, including robotics, medical imaging, terrain navigation and computational geometry. This type of computation on triangular meshes helps to solve different tasks, such as mesh watermarking, shape classification and mesh parametrization. In this work, a priority queue based on a bucketing structure is applied to speed up graph-based methods that approximate shortest geodesic paths on polyhedra. Initially, the problem is stated, some of its properties are discussed and a review of relevant methods is presented. Finally, we describe the proposed method and show several results and comparisons that confirm its benefits.

## 1 INTRODUCTION

The computation of shortest geodesic paths (Chen and Han, 1990; Mitchell et al., 1987; Novotni and Klein, 2002) is an important step in many algorithms that address problems in fields of computer science such as motion planning (Hwang and Ahuja, 1992), object representation and recognition (Hamza and Krim, 2003) and dimensionality reduction (Tenenbaum et al., 2000; Onclinx et al., 2010). More specifically, in the area of computer graphics, where triangle meshes are the standard object representation, which can be considered as polyhedra, geodesic paths provide solution to many diverse problems (Peyré et al., 2010; Bose et al., 2011; Ying et al., 2013; Kamousi et al., 2013; Li et al., 2012), including mesh parameterization (Zigelman et al., 2002), mesh watermarking (Wang et al., 2008), shape matching and classification (Hilaga et al., 2001; Bronstein et al., 2010) and shape retrieval (Rabin et al., 2010).

The problem can be stated as finding the shortest path between two points on the surface of a polyhedron. A shortest geodesic path  $\pi(s, t)$  between  $s$  and  $t$  is defined as a path with minimum Euclidean length among all possible paths joining  $s$  to  $t$ , constrained to lie on the surface of the polyhedron. Moreover, the length of  $\pi(s, t)$  is defined as the sum of the lengths of all segments on the faces which the path traverses.

From a practical point of view, algorithms that compute an exact shortest path are unappealing, as they are fairly complex, numerically unstable and may require an exponential number of bits to per-

form the computation associated with *unfolding* of faces along an edge sequence (Agarwal et al., 2002). These drawbacks have motivated researchers to look into practical approximation algorithms (Aleksandrov et al., 2005), which lead us to the concept of  $\varepsilon$ -approximation. A path  $\pi'(s, t)$  between two points  $s$  and  $t$  is an  $\varepsilon$ -approximation of the shortest path  $\pi(s, t)$  if  $\pi'(s, t)/\pi(s, t) \leq 1 + \varepsilon$ , for  $\varepsilon > 0$ .

Most algorithms for computing geodesic distances and paths handle the *single source* variant of the problem, which seeks to determine shortest paths from a source vertex to all other vertices of the polyhedron. Hence, one can compute the shortest geodesic distance for all pairs of vertices — the *all-pair* problem — by combining the solutions to the single source problem from each vertex; however, this is computationally expensive.

Basically, all algorithms that employ a graph to discretize the paths consist of two stages: building such a graph and computing the shortest geodesic paths. A simple way of building the graph is to consider the input triangular mesh as the graph itself; however, this approach does not assure bounds for the approximation. Several works propose ways to obtain an alternative graph that will guarantee an  $\varepsilon$ -approximation of the optimal solution (Aleksandrov et al., 2005; Aleksandrov et al., 1998). Once the graph is built, the second stage can be performed by executing a shortest path algorithm from any source vertex.

Furthermore, when the triangular mesh does not change over time, the graph needs to be built only once. This characteristic is beneficial to applica-

tions that require computation of paths from multiple sources in an unchanged surface, as in the case of motion planning. For this reason, efficient ways of computing shortest paths on graphs have fundamental importance in methods that approximate the solution for shortest geodesic paths.

In this work, we use an adaptive priority queue to improve graph-based algorithms that approximate the solution of shortest geodesic paths on polyhedra. A standard method for computing shortest paths from a source vertex is to apply Dijkstra's algorithm (Dijkstra, 1959). It is well known that its major cost resides in finding the vertex through which the path has the current lowest cost. To decrease the computational cost for this operation, we propose the use of an adaptive priority queue based on a bucketing structure. In a sense, our approach is similar to the radix heap of Ahuja et al. (Ahuja et al., 1990) — used when all costs are integers of limited size — as it takes advantage of the data being mapped into discrete buckets.

This paper is organized as follows. Section 2 provides an overview of methods applied for solving the shortest geodesic path problem. Section 3 describes the proposed approach. Results and comparisons are shown in Section 4. Conclusions drawn from this work are presented in Section 5.

## 2 RELATED WORK

This section briefly reviews the main algorithms found in the literature that address the problem of computing the shortest geodesic paths between points in a mesh.

Most of these algorithms are based on the single-source approach, which follows the idea of constructing a structure that allows one to obtain the geodesic distance from a fixed source to any point on the surface. It is assumed that the source is a vertex of the polyhedron for, if this is not the case, the source point can act as a new vertex once the face where it lies is triangulated.

Several practical applications require the computation of shortest paths. A two-dimensional instance of the problem is to compute the shortest path between two points on the plane such that it avoids a number of polygonal obstacles. Lozano-Pérez and Wesley (Lozano-Pérez and Wesley, 1979) developed an algorithm based on visibility graphs running in  $O(n^2 \log n)$  time, where  $n$  is the number of vertices in all of the obstacles. An optimal  $O(n \log n)$  time algorithm was described by Hershberger and Suri (Hershberger and Suri, 1993).

The three-dimensional version of the problem is

more complex since obstacles are polyhedra in 3D-space. Here, a shortest path may pass through a vertex of the polyhedron or any of the infinite points along a polyhedral edge. Some algorithms provide an exact solution to the problem of computing shortest paths on a polyhedral surface, whereas others are based on heuristics that produce approximate solutions. For non-convex polyhedra, O'Rourke et al. (O'Rourke et al., 1985) introduced an  $O(n^5)$  time algorithm. Sharir and Schorr (Sharir and Schorr, 1986) proposed an  $O(n^3 \log n)$  algorithm for the convex case, capitalizing on the property that a shortest path on a polyhedron *unfolds* into a straight line. The time complexity was improved by Mitchell et al. (Mitchell et al., 1987), who provided an exact solution to the *single source, all destination* shortest path problem on a triangle mesh. They used a *continuous Dijkstra* method that propagates wavefronts of points from the initial point (source). In their  $O(n^2 \log n)$  time algorithm, each mesh edge is subdivided into a set of intervals in order to perform the exact distance calculation. Their algorithm also works for non-convex polyhedra. Mount (Mount, 1985; Mount, 1986) improved the method by Sharir and Schorr in terms of space and time complexity to  $O(n \log n)$  and  $O(n^2 \log n)$ , respectively.

Chen and Han (Chen and Han, 1990) improved the running time with an  $O(n^2)$  algorithm that also provides an exact solution. Varadarajan and Agarwal (Varadarajan and Agarwal, 2000) presented two algorithms for computing a path on nonconvex polyhedra, one running in  $O(n^{5/3} \log^{5/3} n)$  time and a slightly faster one running in  $O(n^{8/5} \log^{8/5} n)$  time. Kapoor (Kapoor, 1999) developed a complex algorithm for finding shortest paths between pairs of points (*single source, single destination*) on the surface of a three-dimensional polyhedron, running in  $O(n \log^2 n)$  time.

Alternatively, approximation algorithms have also been proposed for the shortest geodesic path problem. Papadimitriou (Papadimitriou, 1985) presented an  $O(n^4(L + \log(n/\epsilon))^2/\epsilon^2)$  time algorithm, where  $L$  is the number of bits of precision in the computation model. Agarwal et al. (Agarwal et al., 2002) presented an algorithm that computes a  $(1 + \epsilon)$ -approximate path on a convex polyhedron in  $O(n \log 1/\epsilon + 1/\epsilon^3)$  time. Kimmel and Sethian (Kimmel and Sethian, 1998) employed a variant of the fast-marching method for computing approximate geodesics on meshes in  $O(n \log n)$  time.

Kanai and Suzuki (Kanai and Suzuki, 2001) proposed an iterative method for calculating an approximate shortest path on a polyhedral surface using a selective refinement of the discrete graph. The refine-

ment inserts Steiner points on edges of the polyhedron and applies Dijkstra's algorithm on the augmented graph. The method is compared to an implementation of the approach developed by Chen and Han (Chen and Han, 1990).

Surazhsky et al. (Surazhsky et al., 2005) presented an exact implementation of the *single source, all destination* algorithm proposed by Mitchell et al. (Mitchell et al., 1987). They also extended the algorithm with a merging operation to obtain fast and approximate solutions for geodesic paths with bounded error.

Aleksandrov et al. (Aleksandrov et al., 1998) proposed an algorithm for computing  $\varepsilon$ -approximation shortest geodesic paths that has better time complexity than the methods described previously. The basic idea of their algorithm is to transform the continuous problem of computing shortest geodesic paths over the surface of a polyhedron into the problem of finding shortest paths in a discrete graph. This is accomplished by inserting Steiner points onto the polyhedron surface and computing shortest paths on a graph whose vertices include the polyhedron vertices as well as the Steiner points. The algorithm computes  $\varepsilon$ -approximate paths in  $O\left(\frac{n}{\varepsilon^2} \log n \log \frac{1}{\varepsilon}\right)$  time.

Aleksandrov et al. (Aleksandrov et al., 2005) presented an approximation algorithm for the single source shortest path problem on weighted polyhedral surfaces. A polyhedral surface  $P$  consists of  $n$  triangular faces, each of which has an associated positive weight. The cost of traversing a face is computed as the traversed Euclidean distance multiplied by the weight of that face. In this algorithm, given a parameter  $\varepsilon$ ,  $0 < \varepsilon < 1$ , the cost of a computed path is at most  $1 + \varepsilon$  times the cost of the corresponding weighted shortest path. The algorithm achieves  $O(C(P) \frac{n}{\sqrt{\varepsilon}} \log \frac{n}{\varepsilon} \log \frac{1}{\varepsilon})$  running time, where  $C(P)$  corresponds to a measure of the input size, which depends on the geometry of  $P$  and the weights of its faces.

### 3 PROPOSED METHOD

In this section, we discuss Dijkstra's algorithm combined with a priority queue based on a bucketing structure with the goal of reducing the computational cost of finding approximations for shortest geodesic paths. First of all, the pseudo-code for a generic version of Dijkstra's algorithm is presented in Algorithm 1, where  $G$  is the input graph (obtained from a triangular mesh),  $Q$  is a list of vertices,  $\mathbf{w}$  is a vector containing the cost of each edge (here, the cost is

the Euclidean length of the edge) and  $s$  is a source vertex. The output of the algorithm consists of a list of distances  $d(s, v)$  between the source vertex  $s$  and each vertex  $v \in Q$  and a set of pointers  $b(v)$  allowing one to reconstruct the path from any vertex  $v$  back to  $s$ . Operations  $\text{Insert}(Q, v)$ ,  $\text{ExtractMin}(Q)$  and  $\text{DecreaseKey}(Q, v)$  will be described in more details along this section.

---

#### Algorithm 1: Dijkstra's algorithm.

---

```

 $d(s, v) \leftarrow \infty \quad \forall v \in G$ 
 $b(v) \leftarrow \emptyset \quad \forall v \in G$ 
 $d(s, s) \leftarrow 0$ 
 $\text{Insert}(Q, s)$  // insert the source vertex  $s$ 
into the list  $Q$ 
while  $Q$  is not empty do
     $u \leftarrow \text{ExtractMin}(Q)$  // remove the vertex
with the smallest distance from the source
    mark  $u$  as visited
    for each edge  $(u, v)$  do
        if  $v$  is not visited then
            if  $d(s, u) + w(u, v) < d(s, v)$  then
                 $d(s, v) \leftarrow d(s, u) + w(u, v)$ 
                 $b(v) \leftarrow u$ 
                if  $v$  is in  $Q$  then
                     $\text{DecreaseKey}(Q, v)$  // update
 $v$ 's position in  $Q$ 
                else  $\text{Insert}(Q, v)$  // insert the
new vertex into the queue
    
```

---

In Dijkstra's algorithm, as a vertex  $v$  is visited, it is assigned a tentative value for  $d(s, v)$ , which is eventually reduced to the final shortest path distance. Each vertex  $v$  inserted into the priority queue  $Q$  is held there until the value of  $d(s, v)$  becomes minimum among vertices in  $Q$ . For the minimum vertex  $u$  among those in  $Q$ , it is known that the value of  $d(s, u)$  cannot be reduced further and is therefore the ultimate shortest path distance. Vertex  $u$  is then removed from  $Q$  and all vertices  $v$  adjacent to  $u$  have their current best estimate  $d(s, v)$  revised and possibly reduced. This process iterates until  $Q$  becomes empty, at which point all vertices have their corresponding shortest path distances correctly determined.

A trade-off exists between the computational cost of two of the main operations over  $Q$ , depending on the choice of data structure used to implement the priority queue.  $\text{ExtractMin}(Q)$  and  $\text{DecreaseKey}(Q, v)$  are the costliest ones, since  $\text{Insert}(Q, v)$  can easily be implemented in constant time. A straightforward implementation as an array of vertices in which  $\text{ExtractMin}(Q)$  searches for the minimum vertex leads Dijkstra's algorithm to an  $O(E + V^2)$  time complex-

ity on a graph with  $V$  vertices and  $E$  edges. A better approach can be attained by implementing  $Q$  as a *regular heap*, as a *Fibonacci heap* (Cormen et al., 2001), or as a *binary heap* (Barbehenn, 1998). In the first case, the overall time complexity becomes  $O((E + V) \log V)$  and in the last two cases  $O(E + V \log V)$ .

In order to further reduce the computational cost due to  $\text{ExtractMin}(Q)$ , our approach creates a priority queue using a bucketing structure indexed according to the distance from the source vertex. In general, a bucketing structure splits a range of values into a finite number of indexed intervals and each bucket contains elements according to their indices.

If we combine the bucketing structure with Dijkstra's algorithm, then each time  $\text{ExtractMin}(Q)$  is called, we are only interested in the first non-empty bucket because it contains the vertex with the smallest distance from the source vertex  $s$ ; therefore, the remaining buckets do not need to be accessed. Hence, a bucketing priority queue can be designed based just on the operations  $\text{Insert}(Q, v)$ ,  $\text{ExtractMin}(Q)$  and  $\text{DecreaseKey}(Q, v)$ , while using the distances from the source vertex as indices. Figure 1 illustrates how the proposed priority queue is employed to select the vertex with the smallest distance from the source vertex  $s$ .

Operation  $\text{Insert}(Q, v)$  that adds a new vertex into the queue takes constant time as it depends only on the index of the vertex and the interval covered by the bucket to be inserted. This operation is shown in Algorithm 2, where  $v$  is the vertex to be inserted,  $w$  is the width of each bucket and  $n$  represents the current number of buckets.

---

**Algorithm 2:** Function  $\text{Insert}(Q, v)$ .

---

```

 $i \leftarrow \lfloor d(s, v) / w \rfloor$ 
// create more buckets according to the path
length  $d(s, v)$ 
if  $i > n$  then
    create  $i - n$  new buckets
     $n \leftarrow i$ 
 $\text{bucket}_i \leftarrow \text{bucket}_i \cup \{v\}$  // insert  $v$  into
bucket  $i$ 

```

---

$\text{ExtractMin}(Q)$  is responsible for locating the bucket containing a vertex with the lowest cost and returning one such vertex. To avoid searching from the first bucket each time  $\text{ExtractMin}(Q)$  is called, it suffices to start the search from the bucket containing the vertex selected in the previous execution of  $\text{ExtractMin}(Q)$ , because the costs increase monotonically in the bucketing structure. This way, the total time spent searching for the correct bucket will be at

most a constant times the number of buckets.

Furthermore, once the right bucket is located, to find the vertex with the smallest cost can be accomplished in time proportional to the number of elements in the bucket.  $\text{ExtractMin}(Q)$  is shown in Algorithm 3.

---

**Algorithm 3:** Function  $\text{ExtractMin}(Q)$ .

---

```

 $i \leftarrow$  index of the first non-empty bucket
 $v \leftarrow$  vertex with the smallest distance in  $\text{bucket}_i$ 
remove  $v$  from  $\text{bucket}_i$ 

```

---

Lastly, function  $\text{DecreaseKey}(Q, v)$  simply determines the index of the new bucket into which  $v$  is to be placed and completes the necessary assignments, which takes constant time per operation. The overall time complexity of the calls to  $\text{DecreaseKey}$  depends on the total number of times this function is executed. In fact, for a particular vertex  $v$ ,  $\text{DecreaseKey}(Q, v)$  might conceivably be called  $O(V)$  times. However, we will show that not only this cannot happen for more than a constant number of vertices but also that the total time complexity of  $\text{DecreaseKey}$  is  $O(V)$ . To see this, since we are dealing only with triangulated meshes that model closed orientable surfaces of bounded genus, consider Euler's Polyhedron Formula relating the number of vertices,  $V$ , edges,  $E$ , faces,  $F$  and the Euler's characteristic,  $\chi$ , of the model

$$V - E + F = \chi \quad (1)$$

Since the mesh is triangulated, we know that  $3F \leq 2E$  and hence  $3F = 3\chi + 3E - 3V \leq 2E$ . Therefore,  $E \leq 3(V - \chi)$ . However

$$\sum_{i=1}^V \deg(v_i) = 2E \leq 6(V - \chi) \quad (2)$$

which implies that the average degree of the vertices of the triangulated mesh is constant. In conclusion, since the maximum number of times that the length of the shortest path to a vertex  $v$  can be revised in Dijkstra's algorithm is  $\deg(v)$ , we conclude that  $\text{DecreaseKey}$  (shown in Algorithm 4) will be called, in total, at most  $O(V)$  times spending constant time per operation.

Using the assumption that the shortest geodesic paths will be computed from multiple source vertices in an unchanged model, we are able to refine the set of parameters used by the priority queue after each shortest paths computation. For this reason, we refer to the proposed method as *adaptive*.

To allow adaptivity in the width of the buckets, we change operation  $\text{Insert}(Q, v)$  in the following manner. For the first computation of the shortest path, we set  $w$

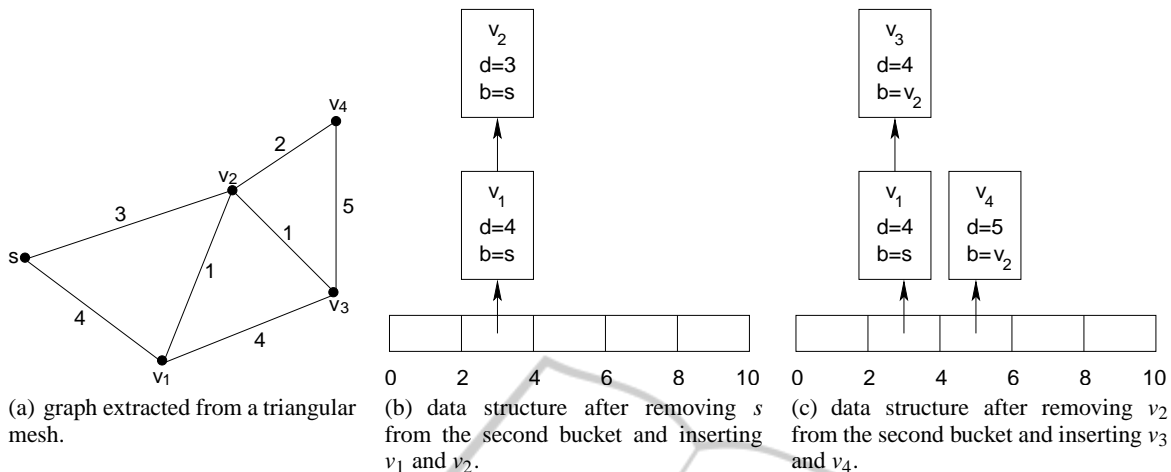


Figure 1: Illustration of the removal and insertion operations for a specific graph.  $d$  denotes the distance from the source vertex  $s$  and  $b$  is a pointer to the previous vertex in the path.

**Algorithm 4:** Function  $\text{DecreaseKey}(Q, v)$ .

```

j ← index of the bucket currently containing v
i ← ⌊d(s,v)/w⌋ // compute the index of
the new bucket for v
bucket_j ← bucket_j - {v} // remove v
from bucket j
bucket_i ← bucket_i ∪ {v} // insert v into
bucket i
    
```

as a fraction of the maximum edge size (used as cost for Dijkstra’s algorithm), whereas  $n$  is initially set to a constant.

After the first shortest paths computation, we refine the input parameters by decreasing the bucket width  $w$  in order to reduce the number of elements in each bucket and hence also the number of comparisons required to find a minimum vertex in a bucket. However, there is a trade-off between the sizes of  $w$  and  $n$ . In order to avoid the overhead of visiting too many buckets, which would happen if we reduced the bucket width indefinitely, we limit  $w$  to be no less than a constant that depends on the mesh at hand. The principle behind this idea is that the number of comparisons decreases by indirectly setting the maximum number of elements in each bucket according to the input data since the variation of edge lengths is mesh dependent.

**4 EXPERIMENTAL RESULTS**

In this section, we show results comparing the proposed approach and other methods for implementing the queue employed by Dijkstra’s algorithm. The re-

sults were obtained on a standard PC featuring an Intel® Core 2 T200 processor, 2 Gbytes of RAM, running Windows® XP operating system.

The parameters used in the experiments are the following:  $n = 5000$  and the fraction of the maximum edge size is set to  $w = 1/c$ , where  $c$  is initially equal to 100. After each distance computation,  $c$  is increased by 100 if the average number of vertices per bucket in the previous iteration is greater than 64.

Table 1 lists the models used in the experiments as well as their numbers of vertices, edges and faces. To obtain realistic results, we chose a set of models so that the number of vertices range from few thousands to over a million.

Table 1: Models used in our experiments.

Model	Vertices	Edges	Faces
Bunny	35,947	105,396	69,451
Armadillo	172,974	527,916	354,944
Virgin	252,470	752,468	500,000
Hand	327,323	981,987	654,666
Dragon	437,645	1,309,057	871,414
Happy Buddha	543,652	1,631,366	1,087,716
Gargo	863,210	2,589,628	1,726,420
Blade	882,954	2,648,340	1,765,388
Amphora	1,317,152	3,950,260	2,633,110

Table 2 shows the number of buckets, average number of elements per bucket and the standard deviation computed for each model. According to the standard deviation, it is possible to observe that the number of elements per bucket is almost constant, which leads to a very small number of comparisons performed by the function  $\text{ExtractMin}(Q)$ .

Table 3 relates the actual number of comparisons performed by the proposed method (second column) with the values of  $E + V$  and  $E + V \log_2 V$ . The num-

Table 2: Number of buckets, average number of elements per bucket and its standard deviation computed for each model used in the experiments.

Model	Number of Buckets	Average Number of Elements per Bucket	Standard Deviation
Bunny	55,077	1.438	0.719
Armadillo	120,562	1.999	1.197
Virgin	186,696	1.939	1.177
Hand	290,708	2.122	1.409
Dragon	298,464	1.933	1.164
Happy Buddha	368,808	1.962	1.209
Gargo	632,312	2.032	1.311
Blade	691,346	1.939	1.195
Amphora	933,554	1.888	1.105

ber of comparisons includes the number of edges, vertices and buckets visited during the entire execution of Dijkstra's algorithm.

 Table 3: Number of comparisons performed by our method and the expected number of comparisons according to different time complexities.  $E + V \log_2 V$  is the expected number of comparisons to Dijkstra's algorithm using Fibonacci heap.  $V$  and  $E$  represent the number of vertices and edges in the graph, respectively.

Model	Number of Comparisons	$E + V$	$E + V \log_2 V$
Bunny	383,427	141,343	649,402
Armadillo	1,850,280	700,890	3,537,697
Virgin	2,682,909	1,004,938	5,283,232
Hand	3,604,024	1,309,310	6,978,660
Dragon	4,632,881	1,746,702	9,510,262
Happy Buddha	5,786,322	2,175,018	11,989,200
Gargo	9,291,953	3,452,838	19,611,569
Blade	9,437,125	3,531,294	20,088,428
Amphora	13,955,707	5,267,412	30,726,630

According to experimental results shown in Figure 2, the method proposed here achieves a number of comparisons between  $E + V \log V$  (due to Fibonacci heap) and  $E + V$  (that is, linear on the number of vertices and edges).

Table 4 shows a comparison between the different implementations of the priority queue used in Dijkstra's algorithm. We compare four methods: a quadratic array-based one; a Fibonacci heap; and our proposed method based on an adaptive bucketing-based heap.

Figure 3 displays the models used in our experiments to solve the *single source, all destination* problem. In the results, we use *hot-to-cold* color ramp, where blue is chosen for the low values of distance, green for middle values and red for high values. For example, on the Bunny, Figure 3(f), the source vertex is on its right ear while its left ear is green since the paths must lie on the surface. Also, the tail is red since it is the farthest body part from the source vertex.

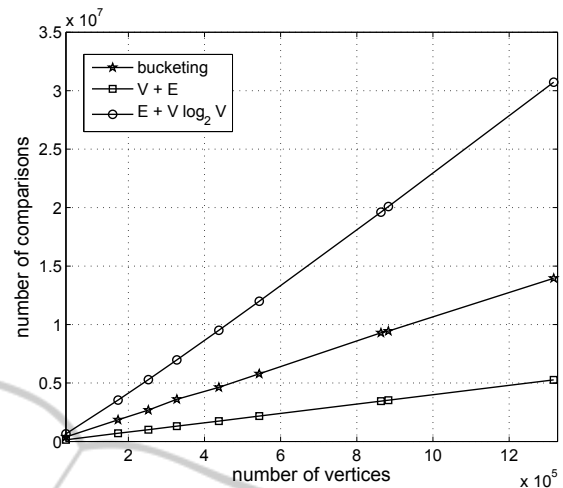

 Figure 2: Graph showing the growth behavior of the number of comparisons performed by the proposed method (bucketing) and of the functions  $E + V$  and  $E + V \log_2 V$ .

Table 4: Time comparison between the various priority queue implementations in Dijkstra's algorithm. The results shown are the average of 150 executions. Preprocessing is the time required to build the graph, which is done only once.

Model	Preprocessing (s)	Array-based (s)	Fibonacci Heap (s)	Bucketing (Heap) (s)
Bunny	0.140	4.453	0.109	<b>0.035</b>
Armadillo	0.891	-	0.622	<b>0.156</b>
Virgin	0.844	-	0.837	<b>0.209</b>
Hand	1.109	-	1.133	<b>0.264</b>
Dragon	1.687	-	1.514	<b>0.397</b>
Happy Buddha	2.141	-	1.992	<b>0.481</b>
Gargo	2.953	-	3.172	<b>0.721</b>
Blade	2.922	-	3.074	<b>0.802</b>
Amphora	4.609	-	4.812	<b>1.287</b>

According to the results shown in Table 4, the proposed method achieves the best computational time on all models considered. On average, it is nearly four times faster than the second best (based on the Fibonacci heap). Furthermore, the results show that the proposed method is suitable to be used in applications requiring a large number of computations of *single source, all destination* shortest geodesic paths.

Although there are no bounds that guarantee an  $\epsilon$ -approximation when directly using the original mesh to create the graph, many applications require only a reasonable estimate of the shortest geodesic path, provided that such estimation is achieved quickly. For instance, by using the original mesh of the Bunny model, our method is able to approximate more than 28 *single source, all destination* shortest geodesic paths per second, while the Fibonacci heap-based method would compute no more than 10.

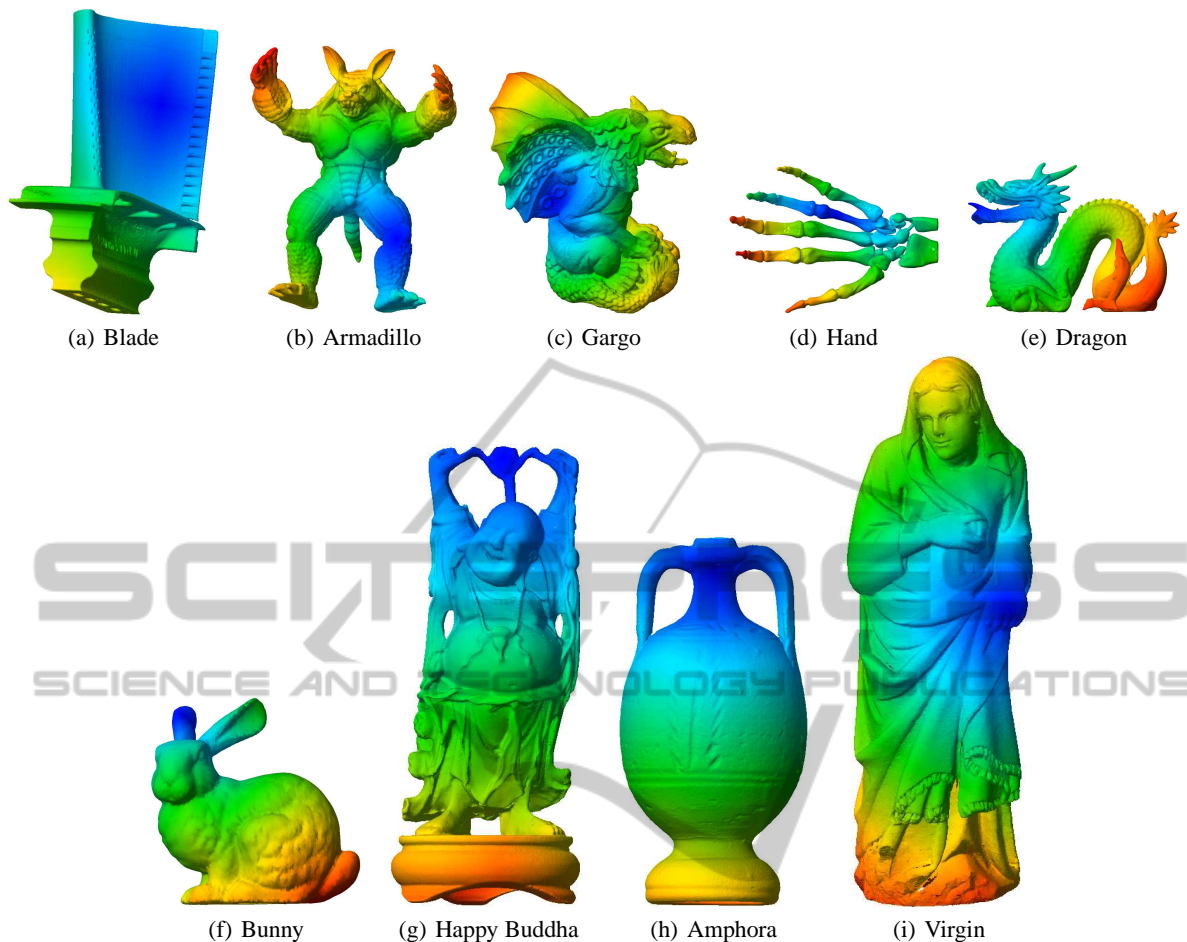


Figure 3: Models used in our experiments to compute approximations of the shortest geodesic paths.

## 5 CONCLUSIONS

Several application areas, such as robotics, medical imaging, terrain navigation and computational geometry, benefit from the computation of shortest geodesic paths, which can be stated as finding the shortest path between two points on the surface of a polyhedron.

In this work, we propose the use of a priority queue based on a bucketing data structure in Dijkstra's algorithm for computing approximations of shortest geodesic paths. The experiments show that our approach is fast and can be used in applications that require approximations for the geodesic distance.

## ACKNOWLEDGEMENTS

This research was supported by grants from: FAPESP, FAPEMIG, CAPES, CNPq #477692/2012-5, #307113/2012-4 and #477457/2013-4.

## REFERENCES

- Agarwal, P., Har-Peled, S., and Karia, M. (2002). Computing Approximate Shortest Paths on Convex Polytopes. *Algorithmica*, 33:227–242.
- Ahuja, R. K., Mehlhorn, K., Orlin, J. B., and Tarjan, R. E. (1990). Faster Algorithms for the Shortest Path Problem. *Journal of the ACM*, 37:213–223.
- Aleksandrov, L., Lanthier, M., Maheshwari, A., and Sack, J.-R. (1998). An  $\epsilon$ -approximation for Weighted Shortest Paths on Polyhedral Surfaces. In *Proc. 6th Scandinavian Workshop on Algorithm Theory - Lecture Notes in Computer Science*, volume 1432, pages 11–22.
- Aleksandrov, L., Maheshwari, A., and Sack, J.-R. (2005). Determining Approximate Shortest Paths on Weighted Polyhedral Surfaces. *Journal of the ACM*, 52(1):25–53.
- Barbehenn, M. (1998). A Note on the Complexity of Dijkstra's Algorithm for Graphs with Weighted Vertices. *IEEE Transactions on Computers*, 47:263.
- Bose, P., Maheshwari, A., Shu, C., and Wührer, S. (2011).

- A Survey of Geodesic Paths on 3D Surfaces. *Computational Geometry*, 44(9):486–498.
- Bronstein, A., Bronstein, M., Kimmel, R., Mahmoudi, M., and Sapiro, G. (2010). A Gromov-Hausdorff Framework with Diffusion Geometry for Topologically-Robust Non-rigid Shape Matching. *International Journal of Computer Vision*, 89:266–286.
- Chen, J. and Han, Y. (1990). Shortest Paths on a Polyhedron. In *Proc. 6th Annual Symposium on Computational Geometry*, pages 360–369.
- Cormen, T. H., Leiserson, C. E., Rivest, R. L., and Stein, C. (2001). *Introduction to Algorithms*. MIT Press and McGraw-Hill, second edition.
- Dijkstra, E. W. (1959). A Note on Two Problems in Connection with Graphs. *Numerische Mathematik*, 1:269–271.
- Hamza, A. B. and Krim, H. (2003). Geodesic Object Representation and Recognition. In *Proc. International Conference on Discrete Geometry for Computer Imagery*, volume 2, pages 378–387.
- Hershberger, J. and Suri, S. (1993). Efficient Computation of Euclidean Shortest Paths in the Plane. In *Proc. 34th Annual Symposium on Foundations of Computer Science*, pages 508–517, Palo Alto, CA, USA.
- Hilaga, M., Shinagawa, Y., Kohmura, T., and Kunit, T. L. (2001). Topology Matching for Fully Automatic Similarity Estimation of 3D Shapes. In *Proc. Conference on Computer Graphics (SIGGRAPH)*, pages 203–212.
- Hwang, Y. K. and Ahuja, N. (1992). Gross Motion Planning: A Survey. *ACM Computer Survey*, 24(3):219–291.
- Kamoussi, P., Lazard, S., Maheshwari, A., and Wuhler, S. (2013). Analysis of Farthest Point Sampling for Approximating Geodesics in a Graph. *arXiv.org*.
- Kanai, T. and Suzuki, H. (2001). Approximate Shortest Path on a Polyhedral Surface and its Applications. *Computer-Aided Design*, 33(11):801–811.
- Kapoor, S. (1999). Efficient Computation of Geodesic Shortest Paths. In *Proc. 31st Annual ACM Symposium on Theory of Computing*, pages 770–779, Atlanta-GA, USA.
- Kimmel, R. and Sethian, J. A. (1998). Computing Geodesic Paths on Manifolds. *Proc. National Academy of Sciences of the United States of America*, 95(15):8431–8435.
- Li, Z., Jin, Y., Jin, X., and Ma, L. (2012). Approximate Straightest Path Computation and its Application in Parameterization. *The Visual Computer*, 28(1):63–74.
- Lozano-Prez, T. and Wesley, M. A. (1979). An Algorithm for Planning Collision-Free Paths among Polyhedral Obstacles. *Communications of the ACM*, 22(10):560–570.
- Mitchell, J. S. B., Mount, D. M., and Papadimitriou, C. H. (1987). The Discrete Geodesic Problem. *SIAM Journal on Computing*, 16(4):647–668.
- Mount, D. (1985). On Finding Shortest Paths in Convex Polyhedra. Technical Report 1495, University of Maryland, Baltimore, USA.
- Mount, D. (1986). Storing the Subdivision of a Polyhedral Surface. In *Second Annual Symposium on Computational Geometry*, pages 150–158.
- Novotni, M. and Klein, R. (2002). Computing Geodesic Distances on Triangular Meshes. In *Proc. 10th International Conference in Central Europe on Computer Graphics*, pages 341–347.
- Onclinx, V., Lee, J., Wertz, V., and Verleysen, M. (2010). Dimensionality Reduction by Rank Preservation. In *International Joint Conference on Neural Networks*, pages 1–8.
- O’Rourke, J., Suri, S., and Booth, H. (1985). Shortest Paths on Polyhedral Surfaces. In *Proc. 2nd Symposium of Theoretical Aspects of Computer Science*, pages 243–254.
- Papadimitriou, C. H. (1985). An Algorithm for Shortest-Path Motion in Three Dimensions. *Information Processing Letters*, 20:259–263.
- Peyré, G., Péchaud, M., Keriven, R., and Cohen, L. D. (2010). Geodesic Methods in Computer Vision and Graphics. *Foundations and Trends in Computer Graphics and Vision*, 5:197–397.
- Rabin, J., Peyré, G., and Cohen, L. D. (2010). Geodesic Shape Retrieval via Optimal Mass Transport. In *Proceedings of the 11th European conference on Computer Vision: Part V*, pages 771–784, Heraklion, Crete, Greece. Springer-Verlag.
- Sharir, M. and Schorr, A. (1986). On Shortest Paths in Polyhedral Spaces. *SIAM Journal on Computing*, 15(1):193–215.
- Surazhsky, V., Surazhsky, T., Kirsanov, D., Gortler, S. J., and Hoppe, H. (2005). Fast Exact and Approximate Geodesics on Meshes. *ACM Transactions on Graphics*, 24(3):553–560.
- Tenenbaum, J. B., de Silva, V., and Langford, J. C. (2000). A Global Geometric Framework for Nonlinear Dimensionality Reduction. *Science*, 290:2319–2323.
- Varadarajan, K. R. and Agarwal, P. K. (2000). Approximating Shortest Paths on a Nonconvex Polyhedron. *SIAM Journal on Computing*, 30(4):1321–1340.
- Wang, K., Lavoue, G., Denis, F., and Baskurt, A. (2008). A Comprehensive Survey on Three-Dimensional Mesh Watermarking. *IEEE Transactions on Multimedia*, 10(8):1513–1527.
- Ying, X., Wang, X., and He, Y. (2013). Saddle Vertex Graph (SVG): A Novel Solution to the Discrete Geodesic Problem. *ACM Transactions on Graphics*, 32(6):170:1–170:12.
- Zigelman, G., Kimmel, R., and Kiryati, N. (2002). Texture Mapping Using Surface Flattening via Multidimensional Scaling. *IEEE Transactions on Visualization and Computer Graphics*, 8(2):198–207.