An Extended Paradefinte Belnap–Dunn Logic that is Embeddable into Classical Logic and Vice Versa

Norihiro Kamide
Department of Information and Electronic Engineering,
Faculty of Science and Engineering, Teikyo University,
Toyosatodai 1-1, Utsunomiya, Tochigi, Japan

Keywords: Paradefinite Logic, Belnap–Bunn Logic, Embedding Theorem, Completeness Theorem, Cut-elimination Theorem.

Abstract: In this study, an extended paradefinite Belnap–Dunn logic (PBD) is introduced as a Gentzen-type sequent calculus. The logic PBD is an extension of Belnap–Dunn logic as well as a modified subsystem of Arieli, Avron, and Zamansky's ideal four-valued paradefinite logic known as 4CC. The logic PBD is formalized on the basis of the idea of De and Omori's characteristic axiom scheme for an extended Belnap–Dunn logic with classical negation (BD+), even though PBD has no classical negation connective but can simulate classical negation. Theorems for syntactically and semantically embedding PBD into a Gentzen-type sequent calculus for classical logic and vice versa are proved. The cut-elimination and completeness theorems for PBD are obtained via these embedding theorems.

1 INTRODUCTION

In this study, a new extended paradefinite Belnap–Dunn logic (PBD) is introduced as a Gentzen-type sequent calculus. This logic is an extension of *Belnap–Dunn logic* (also called *first-degree entailment logic* or *useful four-valued logic*) (Belnap, 1977b; Belnap, 1977a; Dunn, 1976). It is a modified subsystem of *Arieli, Avron, and Zamansky's ideal four-valued paradefinite logic* known as 4CC (Arieli and Avron, 2016; Arieli and Avron, 2017; Arieli et al., 2011). The logic 4CC, which is also an extension of Belnap–Dunn logic, is regarded as a variant of the *logic of logical bilattices* (Arieli and Avron, 1996; Arieli and Avron, 1998). Belnap–Dunn logic and the logic of logical bilattices are well-known to be used as the logical basis for the semantics of logic programming (Fitting, 2002). The proposed logic PBD is also a modification of the logic PL introduced and studied by Kamide and Zohar in (Kamide, 2017; Kamide and Zohar, 2018) as an alternative ideal paradefinite logic embeddable into classical logic and vice versa.

The logic PBD is deemed as a specific type of *paraconsistent logic* (Priest, 2002) with multiple names: it is called *paradefinite logic* by Arieli and Avron (Arieli and Avron, 2016; Arieli and Avron, 2017), *non-alethic logic* by da Costa, and *paranormal logic* by Béziau (Beziau, 2009). Regardless of its name, paradefinite logic incorporates the properties of both *paraconsistency*, which rejects *the principle* $(\alpha\wedge{\sim}\alpha)\rightarrow\beta$ *of explosion*, and *paracompleteness*, which rejects *the law* $\alpha\vee{\sim}\alpha$ *of excluded middle*. Paradefinite logic is known to be appropriate for handling inconsistent and incomplete information, i.e. indefinite information (Arieli and Avron, 2016). Belnap–Dunn logic and 4CC are also paradefinite logics. A good paradefinite logic applicable to the field of computer science has been required.

In this study, we prove theorems for syntactically and semantically embedding PBD into a Gentzen-type sequent calculus LK for classical logic and vice versa. The approach of this study is similar to those presented by Kamide et al. in (Kamide, 2016; Kamide and Shramko, 2017; Kamide, 2017; Kamide and Zohar, 2018) for some paraconsistent and paradefinite logics, including *multilattice logics* introduced by Shramko (Shramko, 2016). We obtain the cut-elimination and completeness theorems for PBD via these embedding theorems. Such an embedding-based proof method has recently been studied, for example in (Kamide, 2015; Kamide, 2016; Kamide and Shramko, 2017; Kamide, 2017; Kamide and Zohar, 2018) to prove the cut-elimination and completeness theorems for some paraconsistent logics.

Kamide, N.
An Extended Paradefinte Belnap–Dunn Logic that is Embeddable into Classical Logic and Vice Versa.
DOI: 10.5220/0007251603770387
In *Proceedings of the 11th International Conference on Agents and Artificial Intelligence (ICAART 2019)*, pages 377-387
ISBN: 978-989-758-350-6

A motivation for developing PBD is to obtain a good *ideal* paradefinite logic that can simulate classical logic in such a way that the underlying logic has *bidirectional* embeddings, i.e., embeddings from the underlying paradefinite logic into classical logic and vice versa. Such a logic is required in application areas that use both paraconsistent (or inconsistency-tolerant) and classical reasoning mechanisms. As in such application areas, we must simultaneously handle indefinite (inconsistent and incomplete) information and definite (consistent and complete) information. Some paraconsistent logics that can simulate classical negation via paraconsistent double negation have recently been studied in (Kamide, 2016; Kamide and Shramko, 2017; Kamide, 2017; Kamide and Zohar, 2018). This work showed that some bidirectional embeddings characterize such logics for representing both indefinite and definite information.

A paradefinite logic called PL, which has such bidirectional embeddings and hence can simulate classical negation, was introduced and studied in (Kamide, 2017; Kamide and Zohar, 2018). The authors proved that the cut-elimination and completeness theorems for PL hold by using the aforementioned embedding-based proof method. Thus, the question considered in this study is that "Is there another logic that has bidirectional embeddings?" We answer this question by developing the new logic PBD. We believe that the existence of such bidirectional embeddings is a plausible condition for an *ideal* paradefinite logic originally proposed in (Arieli et al., 2011). We therefore believe that PBD is a good alternative to ideal paradefinite logic.

The proposed logic PBD has a paraconsistent negation connective $\sim$ and a conflation connective $-$, but has no classical negation connective $\neg$. Some $\{\sim, -, \rightarrow\}$-combined logical inference rules in PBD are formalized on the basis of the idea of De and Omori's characteristic axiom scheme $\sim(\alpha{\rightarrow}\beta) \leftrightarrow \neg{\sim}\alpha \wedge {\sim}\beta$ for the *extended Belnap–Dunn logic with classical negation* (BD+) (De and Omori, 2015). The logic BD+ was shown in (De and Omori, 2015) to be essentially equivalent to *Béziau's four-valued modal logic* PM4N (Beziau, 2011), and *Zaitsev's paraconsistent logic* FDEP (Zaitsev, 2012).

Yet another motivation for developing PBD is to obtain a plausible paradefinite logic that is compatible to the aforementioned well-studied families of extended Belnap–Dunn logics concerned with the characteristic axiom scheme by De and Omori. The aim of this study is therefore to combine the following three ideas: (1) extending Belnap–Dunn logic with classical negation by De and Omori (and hence also by Béziau and Zaitsev); (2) the ideal paradefinite logic by Arieli, Avron, and Zamansky; and (3) constructing a paradefinite logic which has bidirectional embeddings. Based on this aim, we elaborate on the idea of constructing PBD next.

The negated-implication inference rules of PBD just correspond to the axiom scheme $\sim(\alpha{\rightarrow}\beta) \leftrightarrow -\alpha \wedge {\sim}\beta$. This axiom scheme is equivalent to the characteristic axiom scheme mentioned above $\sim(\alpha{\rightarrow}\beta) \leftrightarrow \neg{\sim}\alpha \wedge {\sim}\beta$ by assuming the axiom scheme $\neg{\sim}\alpha \leftrightarrow -\alpha$ as considered implicitly in (Arieli and Avron, 2016; Kamide, 2017; Kamide and Zohar, 2018) for the logic EPL or equivalently 4CC. The logic EPL, which has no $\neg$, does have some logical inference rules that implicitly correspond to the axiom schemes $\neg{\sim}\alpha \leftrightarrow -\alpha$ and $\neg{-}\alpha \leftrightarrow {\sim}\alpha$. The conflated-implication inference rules of PBD just correspond to the axiom scheme $-(\alpha{\rightarrow}\beta) \leftrightarrow {\sim}\alpha \vee -\beta$. This axiom scheme is equivalent to the axiom scheme $-(\alpha{\rightarrow}\beta) \leftrightarrow \neg{-}\alpha \vee -\beta$ (or equivalently $-(\alpha{\rightarrow}\beta) \leftrightarrow -\alpha{\rightarrow}-\beta$) by assuming the axiom scheme $\neg{-}\alpha \leftrightarrow {\sim}\alpha$.

We now provide some comparisons among PBD, PL, 4CC, and EPL. Compared with PBD, the logic PL has the logical inference rules that just correspond to $\sim(\alpha{\rightarrow}\beta) \leftrightarrow \alpha \wedge {\sim}\beta$ and $-(\alpha{\rightarrow}\beta) \leftrightarrow \alpha{\rightarrow}-\beta$ instead of those of PBD. The logic EPL is obtained from PL by adding the initial sequents of the form ${\sim}\alpha, -\alpha \Rightarrow$ and $\Rightarrow {\sim}\alpha, -\alpha$. It was shown in (Kamide and Zohar, 2018) that EPL and 4CC are logically-equivalent. Compared with PBD, the logic EPL does not have the two characteristic properties presented in (Kamide and Zohar, 2018): the *quasi-paraconsistency*, which rejects *the principle* $({\sim}\alpha \wedge -\alpha){\rightarrow}\beta$ *of quasi-explosion*, and the *quasi-paracompleteness*, which rejects *the principle* ${\sim}\alpha \vee -\alpha$ *of quasi-excluded middle*. It can be shown that the quasi-paraconsistency and quasi-paracompleteness hold for PBD and PL. The quasi-paraconsistency and quasi-paracompleteness will be formally introduced and discussed in Section 3.

The structure of this paper is as follows. In Section 2, we introduce PBD and LK and address some basic propositions for PBD. Next, in Section 3, we prove theorems for syntactically embedding PBD into LK and vice versa. We also obtain the cut-elimination theorem for PBD by using the syntactical embedding theorem of PBD into LK. Using the cut-elimination theorem, we obtain the quasi-paraconsistency and quasi-paracompleteness for PBD. In Section 4, we prove theorems for semantically embedding PBD into LK and vice versa. Moreover, we also obtain the completeness theorem with respect to a valuation semantics for PBD by using both the syntactical and semantical embedding theorems from PBD into LK. Finally, in Section 5, we conclude this paper and address some remarks.

2 SEQUENT CALCULUS

Formulas of *ideal paradefinite logic* are constructed from countably many propositional variables by the logical connectives $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication), $\sim$ (paraconsistent negation) and $-$ (conflation). We use small letters $p, q, \ldots$ to denote propositional variables, Greek small letters $\alpha, \beta, \ldots$ to denote formulas, and Greek capital letters $\Gamma, \Delta, \ldots$ to represent finite (possibly empty) sets of formulas. An expression $\sharp\Gamma$ with an unary connective $\sharp$ is used to denote the set $\{\sharp\gamma \mid \gamma \in \Gamma\}$. The symbol $\equiv$ is used to denote the equality of symbols. A *sequent* is an expression of the form $\Gamma \Rightarrow \Delta$. An expression $\alpha \Leftrightarrow \beta$ is used to represent the abbreviation of the sequents $\alpha \Rightarrow \beta$ and $\beta \Rightarrow \alpha$. An expression $L \vdash S$ is used to represent the fact that a sequent S is provable in a sequent calculus L. If L of $L \vdash S$ is clear from the context, we omit L in it. We say that two sequent calculi L_1 and L_2 are *theorem-equivalent* if $\{S \mid L_1 \vdash S\} = \{S \mid L_2 \vdash S\}$. A rule R of inference is said to be *admissible* in a sequent calculus L if the following condition is satisfied: For any instance

$$\frac{S_1 \cdots S_n}{S}$$

of R, if $L \vdash S_i$ for all i, then $L \vdash S$. Moreover, R is said to be *derivable* in L if there is a derivation from $S_1, \cdots, S_n$ to S in L. Note that a rule R of inference is admissible in a sequent calculus L if and only if two sequent calculi L and $L + R$ are theorem-equivalent.

A Gentzen-type sequent calculus PBD for an ideal paradefinite logic is defined as follows.

Definition 2.1 (PBD). *The initial sequents of* PBD *are of the following form, for any propositional variable p,*

$$p \Rightarrow p \qquad \sim p \Rightarrow \sim p \qquad -p \Rightarrow -p.$$

The structural inference rules of PBD *are of the form:*

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \alpha, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi} \text{ (cut)}$$

$$\frac{\Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta} \text{ (we-left)} \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \alpha} \text{ (we-right).}$$

The non-negated logical inference rules of PBD *are of the form:*

$$\frac{\alpha, \beta, \Gamma \Rightarrow \Delta}{\alpha \wedge \beta, \Gamma \Rightarrow \Delta} \text{ (}\wedge\text{left)} \quad \frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \wedge \beta} \text{ (}\wedge\text{right)}$$

$$\frac{\alpha, \Gamma \Rightarrow \Delta \quad \beta, \Gamma \Rightarrow \Delta}{\alpha \vee \beta, \Gamma \Rightarrow \Delta} \text{ (}\vee\text{left)} \quad \frac{\Gamma \Rightarrow \Delta, \alpha, \beta}{\Gamma \Rightarrow \Delta, \alpha \vee \beta} \text{ (}\vee\text{right)}$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \beta, \Sigma \Rightarrow \Pi}{\alpha \rightarrow \beta, \Gamma, \Sigma \Rightarrow \Delta, \Pi} \text{ (}\rightarrow\text{left)} \quad \frac{\alpha, \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \rightarrow \beta} \text{ (}\rightarrow\text{right).}$$

The $\sim$-combined logical inference rules of PBD *are of the form:*

$$\frac{\sim\alpha, \Gamma \Rightarrow \Delta \quad \sim\beta, \Gamma \Rightarrow \Delta}{\sim(\alpha \wedge \beta), \Gamma \Rightarrow \Delta} \text{ (}\sim\wedge\text{left)} \quad \frac{\Gamma \Rightarrow \Delta, \sim\alpha, \sim\beta}{\Gamma \Rightarrow \Delta, \sim(\alpha \wedge \beta)} \text{ (}\sim\wedge\text{right)}$$

$$\frac{\sim\alpha, \sim\beta, \Gamma \Rightarrow \Delta}{\sim(\alpha \vee \beta), \Gamma \Rightarrow \Delta} \text{ (}\sim\vee\text{left)} \quad \frac{\Gamma \Rightarrow \Delta, \sim\alpha \quad \Gamma \Rightarrow \Delta, \sim\beta}{\Gamma \Rightarrow \Delta, \sim(\alpha \vee \beta)} \text{ (}\sim\vee\text{right)}$$

$$\frac{-\alpha, \sim\beta, \Gamma \Rightarrow \Delta}{\sim(\alpha \rightarrow \beta), \Gamma \Rightarrow \Delta} \text{ (}\sim\rightarrow\text{left)} \quad \frac{\Gamma \Rightarrow \Delta, -\alpha \quad \Gamma \Rightarrow \Delta, \sim\beta}{\Gamma \Rightarrow \Delta, \sim(\alpha \rightarrow \beta)} \text{ (}\sim\rightarrow\text{right)}$$

$$\frac{\alpha, \Gamma \Rightarrow \Delta}{\sim\sim\alpha, \Gamma \Rightarrow \Delta} \text{ (}\sim\sim\text{left)} \quad \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, \sim\sim\alpha} \text{ (}\sim\sim\text{right)}$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha}{\sim-\alpha, \Gamma \Rightarrow \Delta} \text{ (}\sim-\text{left)} \quad \frac{\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \sim-\alpha} \text{ (}\sim-\text{right).}$$

The $-$-combined logical inference rules of PBD *are of the form:*

$$\frac{-\alpha, -\beta, \Gamma \Rightarrow \Delta}{-(\alpha \wedge \beta), \Gamma \Rightarrow \Delta} \text{ (}-\wedge\text{left)} \quad \frac{\Gamma \Rightarrow \Delta, -\alpha \quad \Gamma \Rightarrow \Delta, -\beta}{\Gamma \Rightarrow \Delta, -(\alpha \wedge \beta)} \text{ (}-\wedge\text{right)}$$

$$\frac{-\alpha, \Gamma \Rightarrow \Delta \quad -\beta, \Gamma \Rightarrow \Delta}{-(\alpha \vee \beta), \Gamma \Rightarrow \Delta} \text{ (}-\vee\text{left)} \quad \frac{\Gamma \Rightarrow \Delta, -\alpha, -\beta}{\Gamma \Rightarrow \Delta, -(\alpha \vee \beta)} \text{ (}-\vee\text{right)}$$

$$\frac{\sim\alpha, \Gamma \Rightarrow \Delta \quad -\beta, \Gamma \Rightarrow \Delta}{-(\alpha \rightarrow \beta), \Gamma \Rightarrow \Delta} \text{ (}-\rightarrow\text{left)} \quad \frac{\Gamma \Rightarrow \Delta, \sim\alpha, -\beta}{\Gamma \Rightarrow \Delta, -(\alpha \rightarrow \beta)} \text{ (}-\rightarrow\text{right)}$$

$$\frac{\alpha, \Gamma \Rightarrow \Delta}{--\alpha, \Gamma \Rightarrow \Delta} \text{ (}--\text{left)} \quad \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, --\alpha} \text{ (}--\text{right)}$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha}{-\sim\alpha, \Gamma \Rightarrow \Delta} \text{ (}-\sim\text{left)} \quad \frac{\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, -\sim\alpha} \text{ (}-\sim\text{right).}$$

Remark 2.2. *In the following, we assume the symbol $\neg$ as the classical negation connective, although $\neg$ is not included in the language of* PBD.

1. ($\sim\rightarrow$left) *and* ($\sim\rightarrow$right) *just correspond to the Hilbert-style axiom scheme $\sim(\alpha \rightarrow \beta) \leftrightarrow -\alpha \wedge \sim\beta$. This axiom scheme is equivalent to the Hilbert-style axiom scheme $\sim(\alpha \rightarrow \beta) \leftrightarrow \neg\sim\alpha \wedge \sim\beta$ by assuming the Hilbert-style axiom scheme $\neg\sim\alpha \leftrightarrow -\alpha$. The axiom scheme $\sim(\alpha \rightarrow \beta) \leftrightarrow \neg\sim\alpha \wedge \sim\beta$ was introduced by De and Omori for constructing the system* BD+ *of extended Belnap–Dunn logic with classical negation* (De and Omori, 2015). *For the axiom scheme $\neg\sim\alpha \leftrightarrow -\alpha$, see the item 3 below.*
2. ($-\rightarrow$left) *and* ($-\rightarrow$right) *just correspond to the Hilbert-style axiom scheme $-(\alpha \rightarrow \beta) \leftrightarrow \sim\alpha \vee -\beta$. This axiom scheme is equivalent to the Hilbert-style axiom scheme $-(\alpha \rightarrow \beta) \leftrightarrow \neg-\alpha \vee -\beta$ or equivalently $-(\alpha \rightarrow \beta) \leftrightarrow -\alpha \rightarrow -\beta$ by assuming the Hilbert-style axiom scheme $\neg-\alpha \leftrightarrow \sim\alpha$. For the axiom scheme $\neg-\alpha \leftrightarrow \sim\alpha$, see the item 3 below.*

3. *As shown in* (Kamide, 2017; Kamide and Zohar, 2018)*, the following logical inference rules, which implicitly correspond to the above-mentioned Hilbert-style axiom schemes* $\neg{\sim}\alpha \leftrightarrow -\alpha$ *and* $\neg-\alpha \leftrightarrow {\sim}\alpha$*, are admissible in the previously proposed system* EPL*:*

$$\frac{\Gamma \Rightarrow \Delta, -\alpha}{{\sim}\alpha, \Gamma \Rightarrow \Delta}\ ({\sim}\text{left}) \qquad \frac{-\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, {\sim}\alpha}\ ({\sim}\text{right})$$

$$\frac{\Gamma \Rightarrow \Delta, {\sim}\alpha}{-\alpha, \Gamma \Rightarrow \Delta}\ (-\text{left}) \qquad \frac{{\sim}\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, -\alpha}\ (-\text{right}).$$

   However, these rules are not admissible in PBD.

4. *The systems* PL *and* EPL *which were introduced in* (Kamide, 2017; Kamide and Zohar, 2018) *have the following logical inference rules instead of* (${\sim}{\rightarrow}$left), (${\sim}{\rightarrow}$right), ($-{\rightarrow}$left) *and* ($-{\rightarrow}$left)*:*

$$\frac{\alpha, {\sim}\beta, \Gamma \Rightarrow \Delta}{{\sim}(\alpha{\rightarrow}\beta), \Gamma \Rightarrow \Delta}\ ({\sim}{\rightarrow}\text{left}^*) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, {\sim}\beta}{\Gamma \Rightarrow \Delta, {\sim}(\alpha{\rightarrow}\beta)}\ ({\sim}{\rightarrow}\text{right}^*)$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad -\beta, \Sigma \Rightarrow \Pi}{-(\alpha{\rightarrow}\beta), \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (-{\rightarrow}\text{left}^*) \qquad \frac{\alpha, \Gamma \Rightarrow \Delta, -\beta}{\Gamma \Rightarrow \Delta, -(\alpha{\rightarrow}\beta)}\ (-{\rightarrow}\text{right}^*)$$

   which just correspond to the Hilbert-style axiom schemes ${\sim}(\alpha{\rightarrow}\beta) \leftrightarrow \alpha \wedge {\sim}\beta$ *and* $-(\alpha{\rightarrow}\beta) \leftrightarrow \alpha{\rightarrow}-\beta$*. The former axiom scheme is a characteristic one for* Nelson's paraconsistent four-valued logic (Almukdad and Nelson, 1984; Nelson, 1949)*, and the latter axiom scheme is a characteristic one for* connexive logics (Angell, 1962; McCall, 1966; Wansing, 2014).

5. *It was shown in* (Kamide and Zohar, 2018) *that* EPL *is theorem-equivalent to the system* G_{4CC} *for one of the original ideal paradefinite logics,* 4CC*, which was introduced by Arieli et al. in* (Arieli and Avron, 2016; Arieli and Avron, 2017).

Next, we show some basic propositions for PBD.

Proposition 2.3. *Sequents of the form* $\alpha \Rightarrow \alpha$ *for any formula* α *are provable in cut-free* PBD.

Proof. By induction on α. ∎

Proposition 2.4. *The following sequents are provable in cut-free* PBD*:*

1. ${\sim}{\sim}\alpha \Leftrightarrow \alpha$,
2. $--\alpha \Leftrightarrow \alpha$,
3. ${\sim}-\alpha \Leftrightarrow -{\sim}\alpha$,
4. ${\sim}(\alpha \wedge \beta) \Leftrightarrow {\sim}\alpha \vee {\sim}\beta$,
5. ${\sim}(\alpha \vee \beta) \Leftrightarrow {\sim}\alpha \wedge {\sim}\beta$,
6. ${\sim}(\alpha{\rightarrow}\beta) \Leftrightarrow -\alpha \wedge {\sim}\beta$,
7. $-(\alpha \wedge \beta) \Leftrightarrow -\alpha \wedge -\beta$,
8. $-(\alpha \vee \beta) \Leftrightarrow -\alpha \vee -\beta$,
9. $-(\alpha{\rightarrow}\beta) \Leftrightarrow {\sim}\alpha \vee -\beta$.

Proof. By using Proposition 2.3. ∎

For the purpose of showing some embedding theorems, we introduce a Gentzen-type sequent calculus LK for classical logic. Formulas of LK are constructed from countably many propositional variables by logical connectives $\wedge$, $\vee$, $\rightarrow$ and $\neg$ (classical negation).

Definition 2.5 (LK)**.** LK *is obtained from the* $\{{\sim}, -\}$*-free fragment of* PBD *by adding the classical negation inference rules of the form:*

$$\frac{\Gamma \Rightarrow \Delta, \alpha}{\neg\alpha, \Gamma \Rightarrow \Delta}\ (\neg\text{left}) \qquad \frac{\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\alpha}\ (\neg\text{right}).$$

As well-known, the cut-elimination theorem holds for LK (see e.g., (Gentzen, 1969; Takeuti, 2013)).

3 SYNTACTICAL EMBEDDING AND CUT-ELIMINATION

We introduce a translation function from the language of PBD into that of LK, and by using this translation, we show several theorems for embedding PBD into LK.

Definition 3.1. *We fix a set* Φ *of propositional variables, and define the sets* $\Phi^n := \{p^n \mid p \in \Phi\}$ *and* $\Phi^c := \{p^c \mid p \in \Phi\}$ *of propositional variables. The language* $\mathcal{L}_{\text{PBD}}$ *of* PBD *is defined using* $\Phi, \wedge, \vee, \rightarrow, {\sim}$ *and* $-$*. The language* $\mathcal{L}_{\text{LK}}$ *of* LK *is defined using* Φ, Φ^n, Φ^c, $\wedge$, $\vee$, $\rightarrow$ *and* $\neg$*. A mapping* f *from* $\mathcal{L}_{\text{PBD}}$ *to* $\mathcal{L}_{\text{LK}}$ *is defined inductively by:*

1. *For any* $p \in \Phi$, $f(p) := p$, $f({\sim}p) := p^n \in \Phi^n$ *and* $f(-p) := p^c \in \Phi^c$,
2. $f(\alpha \wedge \beta) := f(\alpha) \wedge f(\beta)$,
3. $f(\alpha \vee \beta) := f(\alpha) \vee f(\beta)$,
4. $f(\alpha{\rightarrow}\beta) := f(\alpha){\rightarrow}f(\beta)$,
5. $f({\sim}(\alpha \wedge \beta)) := f({\sim}\alpha) \vee f({\sim}\beta)$,
6. $f({\sim}(\alpha \vee \beta)) := f({\sim}\alpha) \wedge f({\sim}\beta)$,
7. $f({\sim}(\alpha{\rightarrow}\beta)) := f(-\alpha) \wedge f({\sim}\beta)$,
8. $f({\sim}{\sim}\alpha) := f(\alpha)$,
9. $f({\sim}-\alpha) := \neg f(\alpha)$,
10. $f(-(\alpha \wedge \beta)) := f(-\alpha) \wedge f(-\beta)$,
11. $f(-(\alpha \vee \beta)) := f(-\alpha) \vee f(-\beta)$,
12. $f(-(\alpha{\rightarrow}\beta)) := f({\sim}\alpha) \vee f(-\beta)$,
13. $f(--\alpha) := f(\alpha)$,
14. $f(-{\sim}\alpha) := \neg f(\alpha)$.

An expression $f(\Gamma)$ denotes the result of replacing every occurrence of a formula α in Γ by an occurrence of $f(\alpha)$. Analogous notation is used for the other mapping g discussed later.

Remark 3.2. *A similar translation as defined in Definition 3.1 has been used by Gurevich* (Gurevich, 1977), *Rautenberg* (Rautenberg, 1979) *and Vorob'ev* (Vorob'ev, 1952) *to embed Nelson's constructive logic* (Almukdad and Nelson, 1984; Nelson, 1949) *into intuitionistic logic. Some similar translations have also recently been used, for example, in* (Kamide, 2015; Kamide, 2016; Kamide and Shramko, 2017) *to embed some paraconsistent logics into classical logic.*

We now show a weak theorem for syntactically embedding PBD into LK.

Theorem 3.3 (Weak syntactical embedding from PBD into LK)**.** *Let* Γ, Δ *be sets of formulas in* $\mathcal{L}_{\mathrm{PBD}}$, *and* f *be the mapping defined in Definition 3.1.*

1. *If* PBD $\vdash \Gamma \Rightarrow \Delta$, *then* LK $\vdash f(\Gamma) \Rightarrow f(\Delta)$.
2. *If* LK $-$ (cut) $\vdash f(\Gamma) \Rightarrow f(\Delta)$, *then* PBD $-$ (cut) $\vdash \Gamma \Rightarrow \Delta$.

Proof. • (1): By induction on the proofs P of $\Gamma \Rightarrow \Delta$ in PBD. We distinguish the cases according to the last inference of P, and show some cases.

1. Case $\sim p \Rightarrow \sim p$: The last inference of P is of the form: $\sim p \Rightarrow \sim p$ for any $p \in \Phi$. In this case, we obtain LK $\vdash f(\sim p) \Rightarrow f(\sim p)$, i.e., LK $\vdash p^n \Rightarrow p^n$ ($p^n \in \Phi^n$), by the definition of f.
2. Case ($\sim$$-$left): The last inference of P is of the form:
$$\frac{\Gamma \Rightarrow \Delta, \alpha}{\sim-\alpha, \Gamma \Rightarrow \Delta}\ (\sim-\text{left}).$$
By induction hypothesis, we have LK $\vdash f(\Gamma) \Rightarrow f(\Delta), f(\alpha)$. Then, we obtain the required fact:
$$\frac{\begin{array}{c}\vdots\\ f(\Gamma) \Rightarrow f(\Delta), f(\alpha)\end{array}}{\neg f(\alpha), f(\Gamma) \Rightarrow f(\Delta)}\ (\neg\text{left})$$
where $\neg f(\alpha)$ coincides with $f(\sim-\alpha)$ by the definition of f.
3. Case ($\sim$$\rightarrow$left): The last inference of P is of the form:
$$\frac{-\alpha, \sim\beta, \Gamma \Rightarrow \Delta}{\sim(\alpha\rightarrow\beta), \Gamma \Rightarrow \Delta}\ (\sim\rightarrow\text{left}).$$
By induction hypothesis, we have LK $\vdash f(-\alpha), f(\sim\beta), f(\Gamma) \Rightarrow f(\Delta)$. Then, we obtain the required fact:
$$\frac{\begin{array}{c}\vdots\\ f(-\alpha), f(\sim\beta), f(\Gamma) \Rightarrow f(\Delta)\end{array}}{f(-\alpha) \wedge f(\sim\beta), f(\Gamma) \Rightarrow f(\Delta)}\ (\wedge\text{left})$$
where $f(-\alpha) \wedge f(\sim\beta)$ coincides with $f(\sim(\alpha\rightarrow\beta))$ by the definition of f.
4. Case ($\sim$$\rightarrow$right): The last inference of P is of the form:
$$\frac{\Gamma \Rightarrow \Delta, -\alpha \quad \Gamma \Rightarrow \Delta, \sim\beta}{\Gamma \Rightarrow \Delta, \sim(\alpha\rightarrow\beta)}\ (\sim\rightarrow\text{left}).$$
By induction hypothesis, we have LK $\vdash f(\Gamma) \Rightarrow f(\Delta), f(-\alpha)$ and LK $\vdash f(\Gamma) \Rightarrow f(\Delta), f(\sim\beta)$. Then, we obtain the required fact:
$$\frac{\begin{array}{c}\vdots\\ f(\Gamma) \Rightarrow f(\Delta), f(-\alpha)\end{array} \quad \begin{array}{c}\vdots\\ f(\Gamma) \Rightarrow f(\Delta), f(\sim\beta)\end{array}}{f(\Gamma) \Rightarrow f(\Delta), f(-\alpha) \wedge f(\sim\beta)}\ (\wedge\text{right})$$
where $f(-\alpha) \wedge f(\sim\beta)$ coincides with $f(\sim(\alpha\rightarrow\beta))$ by the definition of f.

• (2): By induction on the proofs Q of $f(\Gamma) \Rightarrow f(\Delta)$ in LK $-$ (cut). We distinguish the cases according to the last inference of Q. We show only the following case.

Case ($\wedge$right): The last inference of Q is ($\wedge$right).

1. Subcase (1): The last inference of Q is of the form:
$$\frac{f(\Gamma) \Rightarrow f(\Delta), f(\alpha) \quad f(\Gamma) \Rightarrow f(\Delta), f(\beta)}{f(\Gamma) \Rightarrow f(\Delta), f(\alpha\wedge\beta)}\ (\wedge\text{right})$$
where $f(\alpha\wedge\beta)$ coincides with $f(\alpha) \wedge f(\beta)$ by the definition of f. By induction hypothesis, we have PBD $-$ (cut) $\vdash \Gamma \Rightarrow \Delta, \alpha$ and PBD $-$ (cut) $\vdash \Gamma \Rightarrow \Delta, \beta$. We thus obtain the required fact:
$$\frac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta, \alpha\end{array} \quad \begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta, \beta\end{array}}{\Gamma \Rightarrow \Delta, \alpha\wedge\beta}\ (\wedge\text{right}).$$
2. Subcase (2): The last inference of Q is of the form:
$$\frac{f(\Gamma) \Rightarrow f(\Delta), f(\sim\alpha) \quad f(\Gamma) \Rightarrow f(\Delta), f(\sim\beta)}{f(\Gamma) \Rightarrow f(\Delta), f(\sim(\alpha\vee\beta))}\ (\wedge\text{right})$$
where $f(\sim(\alpha\vee\beta))$ coincides with $f(\sim\alpha) \wedge f(\sim\beta)$ by the definition of f. By induction hypothesis, we have PBD $-$ (cut) $\vdash \Gamma \Rightarrow \Delta, \sim\alpha$ and PBD $-$ (cut) $\vdash \Gamma \Rightarrow \Delta, \sim\beta$. We thus obtain the required fact:
$$\frac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta, \sim\alpha\end{array} \quad \begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta, \sim\beta\end{array}}{\Gamma \Rightarrow \Delta, \sim(\alpha\vee\beta)}\ (\sim\vee\text{right}).$$

3. Subcase (3): The last inference of Q is of the form:

$$\frac{f(\Gamma) \Rightarrow f(\Delta), f(-\alpha) \quad f(\Gamma) \Rightarrow f(\Delta), f(-\beta)}{f(\Gamma) \Rightarrow f(\Delta), f(-(\alpha \wedge \beta))}\ (\wedge\text{right})$$

where $f(-(\alpha \wedge \beta))$ coincides with $f(-\alpha) \wedge f(-\beta)$ by the definition of f. By induction hypothesis, we have PBD $-$ (cut) $\vdash \Gamma \Rightarrow \Delta, -\alpha$ and PBD $-$ (cut) $\vdash \Gamma \Rightarrow \Delta, -\beta$. We thus obtain the required fact:

$$\frac{\begin{array}{c}\vdots \\ \Gamma \Rightarrow \Delta, -\alpha\end{array} \quad \begin{array}{c}\vdots \\ \Gamma \Rightarrow \Delta, -\beta\end{array}}{\Gamma \Rightarrow \Delta, -(\alpha \wedge \beta)}\ (-\wedge\text{right}).$$

4. Subcase (4): The last inference of Q is of the form:

$$\frac{f(\Gamma) \Rightarrow f(\Delta), f(-\alpha) \quad f(\Gamma) \Rightarrow f(\Delta), f(\sim\beta)}{f(\Gamma) \Rightarrow f(\Delta), f(\sim(\alpha \rightarrow \beta))}\ (\wedge\text{right})$$

where $f(\sim(\alpha \rightarrow \beta))$ coincides with $f(-\alpha) \wedge f(\sim\beta)$ by the definition of f. By induction hypothesis, we have PBD $-$ (cut) $\vdash \Gamma \Rightarrow \Delta, -\alpha$ and PBD $-$ (cut) $\vdash \Gamma \Rightarrow \Delta, \sim\beta$. We thus obtain the required fact:

$$\frac{\begin{array}{c}\vdots \\ \Gamma \Rightarrow \Delta, -\alpha\end{array} \quad \begin{array}{c}\vdots \\ \Gamma \Rightarrow \Delta, \sim\beta\end{array}}{\Gamma \Rightarrow \Delta, \sim(\alpha \rightarrow \beta)}\ (\sim\rightarrow\text{right}).$$

■

Using Theorem 3.3 and the cut-elimination theorem for LK, we obtain the following cut-elimination theorem for PBD.

Theorem 3.4 (Cut-elimination for PBD)**.** *The rule* (cut) *is admissible in cut-free* PBD*.*

Proof. Suppose PBD $\vdash \Gamma \Rightarrow \Delta$. Then, we have LK $\vdash f(\Gamma) \Rightarrow f(\Delta)$ by Theorem 3.3 (1), and hence LK $-$ (cut) $\vdash f(\Gamma) \Rightarrow f(\Delta)$ by the cut-elimination theorem for LK. By Theorem 3.3 (2), we obtain PBD $-$ (cut) $\vdash \Gamma \Rightarrow \Delta$. ■

Using Theorem 3.3 and the cut-elimination theorem for LK, we obtain a strong theorem for syntactically embedding PBD into LK.

Theorem 3.5 (Syntactical embedding from PBD into LK)**.** *Let* Γ, Δ *be sets of formulas in* $\mathcal{L}_{\text{PBD}}$*, and* f *be the mapping defined in Definition 3.1.*

1. PBD $\vdash \Gamma \Rightarrow \Delta$ *iff* LK $\vdash f(\Gamma) \Rightarrow f(\Delta)$.
2. PBD $-$ (cut) $\vdash \Gamma \Rightarrow \Delta$ *iff* LK $-$ (cut) $\vdash f(\Gamma) \Rightarrow f(\Delta)$.

Proof. • (1): ($\Longrightarrow$): By Theorem 3.3 (1). ($\Longleftarrow$): Suppose LK $\vdash f(\Gamma) \Rightarrow f(\Delta)$. Then we have LK $-$ (cut) $\vdash f(\Gamma) \Rightarrow f(\Delta)$ by the cut-elimination theorem for LK. We thus obtain PBD $-$ (cut) $\vdash \Gamma \Rightarrow \Delta$ by Theorem 3.3 (2). Therefore we have PBD $\vdash \Gamma \Rightarrow \Delta$.

• (2): ($\Longrightarrow$): Suppose PBD $-$ (cut) $\vdash \Gamma \Rightarrow \Delta$. Then we have PBD $\vdash \Gamma \Rightarrow \Delta$. We then obtain LK $\vdash f(\Gamma) \Rightarrow f(\Delta)$ by Theorem 3.3 (1). Therefore we obtain LK $-$ (cut) $\vdash f(\Gamma) \Rightarrow f(\Delta)$ by the cut-elimination theorem for LK. ($\Longleftarrow$): By Theorem 3.3 (2). ■

By using Theorem 3.5, we can obtain the following theorem.

Theorem 3.6 (Decidability for PBD)**.** PBD *is decidable.*

Proof. By decidability of LK, for each α, it is possible to decide if $f(\alpha)$ is provable in LK. Then, by Theorem 3.5, PBD is also decidable. ■

Using Theorem 3.4, we can show the paraconsistency and quasi-paraconsistency for PBD.

Definition 3.7. *A sequent system L is called* explosive *with respect to a negation-like connective* $\sharp$ *if* $L \vdash \alpha, \sharp\alpha \Rightarrow \beta$ *for any formulas* α *and* β*. A sequent system L is called* paraconsistent *with respect to* $\sharp$ *if L is not explosive with respect to* $\sharp$*. A sequent system L is called* quasi-explosive *with respect to the combination of two different negation-like connectives* $\sharp$ *and* $\natural$ *if* $L \vdash \sharp\alpha, \natural\alpha \Rightarrow \beta$ *for any formulas* α *and* β*. A sequent system L is called* quasi-paraconsistent *with respect to the combination of* $\sharp$ *and* $\natural$ *if L is not quasi-explosive with respect to the combination of* $\sharp$ *and* $\natural$*.*

Theorem 3.8 (Paraconsistency and quasi-paraconsistency for PBD)**.** *We have:*

1. PBD *is paraconsistent with respect to* $\sim$ *and* $-$*.*
2. PBD *is quasi-paraconsistent with respect to the combination of* $\sim$ *and* $-$*.*

Proof. Consider sequents $(p, \sim p \Rightarrow q)$, $(p, -p \Rightarrow q)$ and $(\sim p, -p \Rightarrow q)$ where p and q are distinct propositional variables. Then, the unprovability of these sequents are guaranteed by Theorem 3.4 ■

Using Theorem 3.4, we can also show the paracompleteness and quasi-paracompleteness for PBD.

Definition 3.9. *A sequent system L is called* exclusive *with respect to a negation-like connective* $\sharp$ *if L* $\vdash\ \Rightarrow \alpha, \sharp\alpha$ *for any formula* α*. A sequent system L is called* paracomplete *with respect to* $\sharp$ *if L is not exclusive with respect to* $\sharp$*. A sequent system L is called* quasi-exclusive *with respect to the combination of two different negation-like connectives* $\sharp$ *and* $\natural$ *if* $L \vdash\ \Rightarrow \sharp\alpha, \natural\alpha$ *for any formula* α*. A sequent system L is called* quasi-paracomplete *with respect to the combination of* $\sharp$ *and* $\natural$ *if L is not quasi-exclusive with respect to the combination of* $\sharp$ *and* $\natural$*.*

Theorem 3.10 (Paracompleteness and quasi-paracompleteness for PBD)**.** *We have:*

1. PBD *is paracomplete with respect to* $\sim$ *and* $-$.
2. PBD *is quasi-paracomplete with respect to the combination of* $\sim$ *and* $-$.

Proof. Consider sequents $(\Rightarrow p, \sim p)$, $(\Rightarrow p, -p)$ and $(\Rightarrow \sim p, -p)$ where p is a propositional variable. Then, the unprovability of these sequents are guaranteed by Theorem 3.4 ∎

Remark 3.11. *The quasi-paraconsistency and quasparacompleteness do not hold for* EPL (Kamide and Zohar, 2018) *or equivalently* 4CC (Arieli and Avron, 2016)*, since* EPL *has the initial sequents of the form* $\sim\alpha, -\alpha \Rightarrow$ *and* $\Rightarrow \sim\alpha, -\alpha$*. On the other hand, these properties hold for the logic* PL *which was introduced and studied in* (Kamide, 2017; Kamide and Zohar, 2018).

Next, we introduce a translation function from the language of LK into that of PBD, and by using this translation, we show some theorems for embedding LK into PBD.

Definition 3.12. *Let* $\mathcal{L}_{\text{PBD}}$ *and* $\mathcal{L}_{\text{LK}}$ *be the languages defined in Definition 3.1. A mapping* g *from* $\mathcal{L}_{\text{LK}}$ *to* $\mathcal{L}_{\text{PBD}}$ *is defined inductively by:*

1. *For any* $p \in \Phi$, *any* $p^n \in \Phi^n$ *and any* $p^c \in \Phi^c$, $g(p) := p$, $g(p^n) := \sim p$ *and* $g(p^c) := -p$,
2. $g(\alpha \wedge \beta) := g(\alpha) \wedge g(\beta)$,
3. $g(\alpha \vee \beta) := g(\alpha) \vee g(\beta)$,
4. $g(\alpha \rightarrow \beta) := g(\alpha) \rightarrow g(\beta)$,
5. $g(\neg\alpha) := \sim - g(\alpha)$.

Theorem 3.13 (Weak syntactical embedding from LK into PBD)**.** *Let* Γ, Δ *be sets of formulas in* $\mathcal{L}_{\text{LK}}$, *and* g *be the mapping defined in Definition 3.12.*

1. *If* LK $\vdash \Gamma \Rightarrow \Delta$, *then* PBD $\vdash g(\Gamma) \Rightarrow g(\Delta)$.
2. *If* PBD $-$ (cut) $\vdash g(\Gamma) \Rightarrow g(\Delta)$, *then* LK $-$ (cut) $\vdash \Gamma \Rightarrow \Delta$.

Proof. • (1): By induction on the proofs P of $\Gamma \Rightarrow \Delta$ in LK. We distinguish the cases according to the last inference of P, and show only the following cases.

1. Case $p^\star \Rightarrow p^\star$ with $\star \in \{n, c\}$: The last inference of P is of the form: $p^\star \Rightarrow p^\star$ for any $p^\star \in \Phi^\star$ with $\star \in \{n, c\}$. In this case, we obtain PBD $\vdash g(p^n) \Rightarrow g(p^n)$ and PBD $\vdash g(p^c) \Rightarrow g(p^c)$ i.e., PBD $\vdash \sim p \Rightarrow \sim p$ and PBD $\vdash -p \Rightarrow -p$, by the definition of g.
2. Case ($\neg$left): The last inference of P is of the form:
$$\frac{\Gamma \Rightarrow \Delta, \alpha}{\neg\alpha, \Gamma \Rightarrow \Delta}\ (\neg\text{left})$$
By induction hypothesis, we have PBD $\vdash g(\Gamma) \Rightarrow g(\Delta), g(\alpha)$. We then obtain the required fact:
$$\frac{\vdots \atop g(\Gamma) \Rightarrow g(\Delta), g(\alpha)}{\sim - g(\alpha), g(\Gamma) \Rightarrow g(\Delta)}\ (\sim-\text{left})$$
where $\sim - g(\alpha)$ coincides with $g(\neg\alpha)$ by the definition of g.

• (2): By induction on the proofs Q of $g(\Gamma) \Rightarrow g(\Delta)$ in PBD $-$ (cut). We distinguish the cases according to the last inference of Q, and show only the following cases.

1. Case ($\sim-$left): The last inference of Q is of the form:
$$\frac{g(\Gamma) \Rightarrow g(\Delta), g(\alpha)}{\sim - g(\alpha), g(\Gamma) \Rightarrow g(\Delta)}\ (\sim-\text{left})$$
where $\sim - g(\alpha)$ coincides with $g(\neg\alpha)$ by the definition of g. By induction hypothesis, we have LK $-$ (cut) $\vdash \Gamma \Rightarrow \Delta, \alpha$. We thus obtain the required fact:
$$\frac{\vdots \atop \Gamma \Rightarrow \Delta, \alpha}{\neg\alpha, \Gamma \Rightarrow \Delta}\ (\neg\text{left}).$$
2. Case ($\wedge$right): The last inference of Q is of the form:
$$\frac{g(\Gamma) \Rightarrow g(\Delta), g(\alpha) \quad g(\Gamma) \Rightarrow g(\Delta), g(\beta)}{g(\Gamma) \Rightarrow g(\Delta), g(\alpha) \wedge g(\beta)}\ (\wedge\text{right})$$
where $g(\alpha) \wedge g(\beta)$ coincides with $g(\alpha \wedge \beta)$ by the definition of g. By induction hypothesis, we have LK $-$ (cut) $\vdash \Gamma \Rightarrow \Delta, \alpha$ and LK $-$ (cut) $\vdash \Gamma \Rightarrow \Delta, \beta$. We thus obtain the required fact:
$$\frac{\vdots \atop \Gamma \Rightarrow \Delta, \alpha \quad \vdots \atop \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \wedge \beta}\ (\wedge\text{right}).$$

∎

Theorem 3.14 (Syntactical embedding from LK into PBD)**.** *Let* Γ, Δ *be sets of formulas in* $\mathcal{L}_{\text{LK}}$, *and* g *be the mapping defined in Definition 3.12.*

1. LK $\vdash \Gamma \Rightarrow \Delta$ *iff* PBD $\vdash g(\Gamma) \Rightarrow g(\Delta)$.
2. LK $-$ (cut) $\vdash \Gamma \Rightarrow \Delta$ *iff* PBD $-$ (cut) $\vdash g(\Gamma) \Rightarrow g(\Delta)$.

Proof. By using Theorems 3.13 and 3.4. Similar to Theorem 3.5. ∎

4 SEMANTICAL EMBEDDING AND COMPLETENESS

We now introduce a valuation semantics for PBD by defining the valuation function on the two-element set of classical truth-values.

Definition 4.1 (Semantics for PBD)**.** *Let Φ be the set of all propositional variables, $\Phi^{\sim}$ be the set $\{\sim p \mid p \in \Phi\}$ and Φ^{-} be the set $\{-p \mid p \in \Phi\}$. A* paraconsistent valuation *v^* is a mapping from $\Phi \cup \Phi^{\sim} \cup \Phi^{-}$ to the set $\{t, f\}$ of truth values. The paraconsistent valuation v^* is extended to the mapping from the set of all formulas to $\{t, f\}$ by the following clauses.*

1. $v^*(\alpha \wedge \beta) = t$ *iff* $v^*(\alpha) = t$ *and* $v^*(\beta) = t$,
2. $v^*(\alpha \vee \beta) = t$ *iff* $v^*(\alpha) = t$ *or* $v^*(\beta) = t$,
3. $v^*(\alpha \rightarrow \beta) = t$ *iff* $v^*(\alpha) = f$ *or* $v^*(\beta) = t$,
4. $v^*(\sim(\alpha \wedge \beta)) = t$ *iff* $v^*(\sim\alpha) = t$ *or* $v^*(\sim\beta) = t$,
5. $v^*(\sim(\alpha \vee \beta)) = t$ *iff* $v^*(\sim\alpha) = t$ *and* $v^*(\sim\beta) = t$,
6. $v^*(\sim(\alpha \rightarrow \beta)) = t$ *iff* $v^*(-\alpha) = t$ *and* $v^*(\sim\beta) = t$,
7. $v^*(\sim\sim\alpha) = t$ *iff* $v^*(\alpha) = t$,
8. $v^*(\sim-\alpha) = t$ *iff* $v^*(\alpha) = f$,
9. $v^*(-(\alpha \wedge \beta)) = t$ *iff* $v^*(-\alpha) = t$ *and* $v^*(-\beta) = t$,
10. $v^*(-(\alpha \vee \beta)) = t$ *iff* $v^*(-\alpha) = t$ *or* $v^*(-\beta) = t$,
11. $v^*(-(\alpha \rightarrow \beta)) = t$ *iff* $v^*(\sim\alpha) = t$ *or* $v^*(-\beta) = t$,
12. $v^*(--\alpha) = t$ *iff* $v^*(\alpha) = t$,
13. $v^*(-\sim\alpha) = t$ *iff* $v^*(\alpha) = f$.

A formula α is called PBD-valid *iff $v^*(\alpha) = t$ holds for all paraconsistent valuation v^*.*

For the purpose of showing some semantical embedding theorems, we present the standard two-valued semantics for LK.

Definition 4.2 (Semantics for LK)**.** *A valuation v is a mapping from the set of all propositional variables to the set $\{t, f\}$ of truth values. The valuation v is extended to the mapping from the set of all formulas to $\{t, f\}$ by the following clauses.*

1. $v(\alpha \wedge \beta) = t$ *iff* $v(\alpha) = t$ *and* $v(\beta) = t$,
2. $v(\alpha \vee \beta) = t$ *iff* $v(\alpha) = t$ *or* $v(\beta) = t$,
3. $v(\alpha \rightarrow \beta) = t$ *iff* $v(\alpha) = f$ *or* $v(\beta) = t$,
4. $v(\neg\alpha) = t$ *iff* $v(\alpha) = f$.

A formula α is called LK-valid *iff $v(\alpha) = t$ holds for all valuation v.*

The following completeness theorem holds for LK: For any formula α, LK $\vdash\ \Rightarrow \alpha$ iff α is LK-valid.

Next, we show a theorem for semantically embedding PBD into LK.

Lemma 4.3. *Let f be the mapping defined in Definition 3.1. For any paraconsistent valuation v^*, we can construct a valuation v such that for any formula α, $v^*(\alpha) = t$ iff $v(f(\alpha)) = t$.*

Proof. Let Φ be a set of propositional variables, and for each $\star \in \{n, c\}$, let $\Phi^{\star}$ be the set $\{p^{\star} \mid p \in \Phi\}$ of propositional variables. Suppose that v^* is a paraconsistent valuation. Suppose that v is a mapping from $\Phi \cup \Phi^n \cup \Phi^c$ to $\{t, f\}$ such that

1. $v^*(p) = t$ iff $v(p) = t$,
2. $v^*(\sim p) = t$ iff $v(p^n) = t$,
3. $v^*(-p) = t$ iff $v(p^c) = t$.

Then, the lemma is proved by induction on α.

• Base step:

1. Case $\alpha \equiv p$ where p is a propositional variable: $v^*(p) = t$ iff $v(p) = t$ (by the assumption) iff $v(f(p)) = t$ (by the definition of f).
2. Case $\alpha \equiv \sim p$ where p is a propositional variable: $v^*(\sim p) = t$ iff $v(p^n) = t$ (by the assumption) iff $v(f(\sim p)) = t$ (by the definition of f).
3. Case $\alpha \equiv -p$ where p is a propositional variable: Similar to the above case.

• Induction step: We show some cases.

1. Case $\alpha \equiv \sim\sim\beta$: $v^*(\sim\sim\beta) = t$ iff $v^*(\beta) = t$ iff $v(f(\beta)) = t$ (by induction hypothesis) iff $v(f(\sim\sim\beta)) = t$ (by the definition of f).
2. Case $\alpha \equiv \sim(\beta \rightarrow \gamma)$: $v^*(\sim(\beta \rightarrow \gamma)) = t$ iff $v^*(-\beta) = t$ and $v^*(\sim\gamma) = t$ iff $v(f(-\beta)) = t$ and $v(f(\sim\gamma)) = t$ (by induction hypothesis) iff $v(f(-\beta) \wedge f(\sim\gamma)) = t$ iff $v(f(\sim(\beta \rightarrow \gamma))) = t$ (by the definition of f).
3. Case $\alpha \equiv \sim-\beta$: $v^*(\sim-\beta) = t$ iff $v^*(\beta) = f$ iff $v(f(\beta)) = f$ (by induction hypothesis) iff $v(\neg f(\beta)) = t$ iff $v(f(\sim-\beta)) = t$ (by the definition of f).
4. Case $\alpha \equiv -(\beta \rightarrow \gamma)$: $v^*(-(\beta \rightarrow \gamma)) = t$ iff $v^*(\sim\beta) = t$ or $v^*(-\gamma) = t$ iff $v(f(\sim\beta)) = t$ or $v(f(-\gamma)) = t$ (by induction hypothesis) iff $v(f(\sim\beta) \vee f(-\gamma)) = t$ iff $v(f(-(\beta \rightarrow \gamma))) = t$ (by the definition of f). ∎

Lemma 4.4. *Let f be the mapping defined in Definition 3.1. For any valuation v, we can construct a paraconsistent valuation v^* such that for any formula α, $v(f(\alpha)) = t$ iff $v^*(\alpha) = t$.*

Proof. Similar to the proof of Lemma 4.3. ∎

Theorem 4.5 (Semantical embedding from PBD into LK)**.** *Let f be the mapping defined in Definition 3.1. For any formula α, α is PBD-valid iff $f(\alpha)$ is LK-valid.*

Proof. By Lemmas 4.3 and 4.4. ∎

Theorem 4.6 (Completeness for PBD)**.** *For any formula α, PBD $\vdash\ \Rightarrow \alpha$ iff α is PBD-valid.*

Proof. We have: PBD $\vdash\ \Rightarrow \alpha$ iff LK $\vdash\ \Rightarrow f(\alpha)$ (by Theorem 3.5) iff $f(\alpha)$ is LK-valid (by the completeness theorem for LK) iff α is PBD-valid (by Theorem 4.5). ∎

Next, we show a theorem for semantically embedding LK into PBD.

Lemma 4.7. *Let g be the mapping defined in Definition 3.12. For any valuation v, we can construct a paraconsistent valuation v^* such that for any formula α, $v(\alpha) = t$ iff $v^*(g(\alpha)) = t$.*

Proof. Let Φ be a set of propositional variables, and for each $\star \in \{n, c\}$, let $\Phi^\star$ be the set $\{p^\star \mid p \in \Phi\}$ of propositional variables. Suppose that v^* is a paraconsistent valuation. Suppose that v is a mapping from $\Phi \cup \Phi^n \cup \Phi^c$ to $\{t, f\}$ such that

1. $v^*(p) = t$ iff $v(p) = t$,
2. $v^*(\sim p) = t$ iff $v(p^n) = t$,
3. $v^*(-p) = t$ iff $v(p^c) = t$.

Then, the lemma is proved by induction on α.

• Base step:

1. Case $\alpha \equiv p$ where p is a propositional variable: $v(p) = t$ iff $v^*(p) = t$ (by the assumption) iff $v^*(g(p)) = t$ (by the definition of g).
2. Case $\alpha \equiv p^n$ where p is a propositional variable: $v(p^n) = t$ iff $v^*(\sim p) = t$ (by the assumption) iff $v^*(g(p^n)) = t$ (by the definition of g).
3. Case $\alpha \equiv p^c$ where p is a propositional variable: Similar to the above case.

• Induction step: We show only the following case,

Case $\alpha \equiv \neg\beta$: $v(\neg\beta) = t$ iff $v(\beta) = f$ iff $v^*(g(\beta)) = f$ (by induction hypothesis) iff $v^*(\sim -g(\beta)) = t$ iff $v^*(g(\neg\beta)) = t$ (by the definition of g). ∎

Lemma 4.8. *Let g be the mapping defined in Definition 3.12. For any paraconsistent valuation v^*, we can construct a valuation v such that for any formula α, $v^*(g(\alpha)) = t$ iff $v(\alpha) = t$.*

Proof. Similar to the proof of Lemma 4.7. ∎

Theorem 4.9 (Semantical embedding from LK into PBD)**.** *Let g be the mapping defined in Definition 3.12. For any formula α, α is LK-valid iff $g(\alpha)$ is PBD-valid.*

Proof. By Lemmas 4.7 and 4.8. ∎

5 CONCLUDING REMARKS

In this study, we introduced a new extended paradefinite Belnap–Dunn logic (PBD) as a Gentzen-type sequent calculus. This logic is a modified subsystem of Arieli, Avron and Zamansky's ideal four-valued paradefinite logic 4CC. We proved the theorems for syntactically and semantically embedding PBD into LK and vice versa. We then obtained the cut-elimination and completeness theorems for PBD via these embedding theorems. The theorems presented were proved using the same methods as shown in (Kamide and Shramko, 2017; Kamide and Zohar, 2018).

Next, we show some motivations for introducing PBD from the point of view of computer science. Combining Belnap–Dunn logic (paradefinite logic) with classical negation is regarded as an important issue in the field of computer science. Descriptions of both indefinite (or inconsistent) information, which is described by the paraconsistent negation connective $\sim$ in Belnap–Dunn logic (paradefinite logic), and definite (consistent) information, which is described by the classical negation connective $\neg$ (which can be defined as $\sim -$ in PBD) in classical logic, are requirements for appropriately handling certain computer science applications. Indeed, both the negations have been used in many computer science applications such as logic programming and automated theorem proving. Thus, using a combined logic (such as PBD) with the paraconsistent and classical negations, we can naturally handle these applications. For some recent developments of such applications using paraconsistent negation, see e.g., (Ciucci and Dubois, 2017) wherein a logical framework was proposed for handling multi-source inconsistent information. For a recent purely theoretical development of extensions of Belnap–Dunn logic, see (Albuquerque et al., 2017) wherein some super-Belnap logics was studied from an algebraic point of view.

Finally, we show that some additional results can be obtained for PBD and its first-order extension FPBD. By using the same embedding-based method proposed and used in (Kamide, 2015; Kamide and Shramko, 2017), we can obtain a modified Craig interpolation theorem for PBD, which was also shown in (Kamide, 2015; Kamide and Shramko, 2017) for the other logics. As a corollary of this theorem, we can also obtain the Maksimova principle of variable separation for PBD.

The expression $V(\alpha)$ denotes the set of all propositional variables occurring in α.

Theorem 5.1 (Modified Craig interpolation for PBD)**.** *Suppose* PBD $\vdash \alpha \Rightarrow \beta$ *for any formulas α and β. If $V(\alpha) \cap V(\beta) \neq \emptyset$, then there exists a formula γ such that*

1. PBD $\vdash \alpha \Rightarrow \gamma$ *and* PBD $\vdash \gamma \Rightarrow \beta$,
2. $V(\gamma) \subseteq V(\alpha) \cap V(\beta)$.

If $V(\alpha) \cap V(\beta) = \emptyset$, then

3. PBD $\vdash \Rightarrow \sim -\alpha$ *or* PBD $\vdash \Rightarrow \beta$.

As a corollary, we can obtain the following Maksimova principle of variable separation.

Theorem 5.2 (Maksimova principle for PBD)**.** *Suppose $V(\alpha_1,\alpha_2) \cap V(\beta_1,\beta_2) \neq \emptyset$ for any formulas $\alpha_1, \alpha_2, \beta_1$ and β_2. If* PBD $\vdash \alpha_1 \wedge \beta_1 \Rightarrow \alpha_2 \vee \beta_2$, *then either* PBD $\vdash \alpha_1 \Rightarrow \alpha_2$ *or* PBD $\vdash \beta_1 \Rightarrow \beta_2$.

We can also introduce a first-order extension, FPBD, of PBD, as well as its valuation semantics in a natural way. Thus, we can show the theorems for syntactically and semantically embedding FPBD into a Gentzen-type sequent calculus FLK for first-order classical logic. By using these embedding theorems, we can obtain the cut-elimination, completeness, modified Craig interpolation, and Maksimova separation theorems for FPBD.

ACKNOWLEDGEMENTS

We would like to thank the anonymous referees for their valuable comments and suggestions. This research was supported by JSPS KAKENHI Grant Numbers JP18K11171, JP16KK0007, and JSPS Core-to-Core Program (A. Advanced Research Networks). This research has also been supported by the Kayamori Foundation of Informational Science Advancement.

REFERENCES

Albuquerque, H., Prenosil, A., and Rivieccio, U. (2017). An algebraic view of super-belnap logics. *Studia Logica*, 105(6):1051–1086.

Almukdad, A. and Nelson, D. (1984). Constructible falsity and inexact predicates. *Journal of Symbolic Logic*, 49:231–233.

Angell, R. (1962). A propositional logics with subjunctive conditionals. *Journal of Symbolic Logic*, 27:327–343.

Arieli, O. and Avron, A. (1996). Reasoning with logical bilattices. *Journal of Logic, Language and Information*, 5:25–63.

Arieli, O. and Avron, A. (1998). The value of the four values. *Artificial Intelligence*, 102 (1):97–141.

Arieli, O. and Avron, A. (2016). Minimal paradefinite logics for reasoning with incompleteness and inconsistency. *Proceedings of the 1st International Conference on Formal Structures for Computation and Deduction (FSCD), Leibniz International Proceedings in Informatics*, 52:7:1–7:15.

Arieli, O. and Avron, A. (2017). Four-valued paradefinitie logics. *Studia Logica*, 105 (6):1087–1122.

Arieli, O., Avron, A., and Zamansky, A. (2011). Ideal paraconsistent logics. *Studia Logica*, 99 (1-3):31–60.

Belnap, N. (1977a). *How a computer should think, in: in: Contemporary Aspects of Philosophy, (G. Ryle ed.), pp. 30-56*. Oriel Press, Stocksfield.

Belnap, N. (1977b). *A useful four-valued logic in: J.M. Dunn and G. Epstein (eds.), Modern Uses of Multiple-Valued Logic, pp. 5-37*. Reidel, Dordrecht.

Beziau, J.-Y. (2009). *Bivalent semantics for De Morgan logic (The uselessness of four-valuedness), In W.A. Carnieli, M.E. Coniglio and I.M. D'Ottaviano, editors, The many sides of logic, pp. 391-402*. College Publications.

Beziau, J.-Y. (2011). A new four-valued approach to modal logic. *Logique et Analyse*, 54 (213):109–121.

Ciucci, D. and Dubois, D. (2017). A two-tiered propositional framework for handling multisource inconsistent information. In *Symbolic and Quantitative Approaches to Reasoning with Uncertainty - 14th European Conference, ECSQARU 2017, Lugano, Switzerland, July 10-14, 2017, Proceedings*, pages 398–408.

De, M. and Omori, H. (2015). Classical negation and expansions of belnap–dunn logic. *Studia Logica*, 103 (4):825–851.

Dunn, J. (1976). Intuitive semantics for first-degree entailment and 'coupled trees'. *Philosophical Studies*, 29 (3):149–168.

Fitting, M. (2002). Fixpoint semantics for logic programming a survey. *Theoretical Computer Science*, 278 (1-2):25–51.

Gentzen, G. (1969). *Collected papers of Gerhard Gentzen, M.E. Szabo, ed., Studies in logic and the foundations of mathematics*. North-Holland (English translation).

Gurevich, Y. (1977). Intuitionistic logic with strong negation. *Studia Logica*, 36:49–59.

Kamide, N. (2015). Trilattice logic: an embedding-based approach. *Journal of Logic and Computation*, 25 (3):581–611.

Kamide, N. (2016). Paraconsistent double negation that can simulate classical negation. *Proceedings of the 46th IEEE International Symposium on Multiple-Valued Logic*, pages 131–136.

Kamide, N. (2017). Extending ideal paraconsistent four-valued logic. *Proceedings of the 47th IEEE International Symposium on Multiple-Valued Logic*, pages 49–54.

Kamide, N. and Shramko, Y. (2017). Embedding from multilattice logic into classical logic and vice versa. *Journal of Logic and Computation*, 27 (5):1549–1575.

Kamide, N. and Zohar, Y. (2018). Yet another paradefinite logic: The role of conflation. *Logic Journal of the IGPL*, Published onlin first.

McCall, S. (1966). Connexive implication. *Journal of Symbolic Logic*, 31:415–433.

Nelson, D. (1949). Constructible falsity. *Journal of Symbolic Logic*, 14:16–26.

Priest, G. (2002). *Paraconsistent logic, Handbook of Philosophical Logic (Second Edition), Vol. 6, D. Gabbay and F. Guenthner (eds.)*. Kluwer Academic Publishers, Dordrecht, pp. 287-393.

Rautenberg, W. (1979). *Klassische und nicht-klassische Aussagenlogik*. Vieweg, Braunschweig.

Shramko, Y. (2016). Truth, falsehood, information and beyond: the american plan generalized, in: K. bimbo (ed.), j. michael dunn on information based logics. *Outstanding Contributions to Logic*, 8:191–212.

Takeuti, G. (2013). *Proof theory (second edition)*. Dover Publications, Inc. Mineola, New York.

Vorob'ev, N. (1952). A constructive propositional calculus with strong negation (in Russian). *Doklady Akademii Nauk SSR*, 85:465–468.

Wansing, H. (2014). Connexive logic. *Stanford Encyclopedia of Philosophy, http://plato.stanford.edu/entries/logic-connexive/*.

Zaitsev, D. (2012). *Generalized relevant logic and models of reasoning*. Moscow State Lomonosov University doctoral dissertation.