Construction of a Complete Lyapunov Function using Quadratic Programming

Peter Giesl[1], Carlos Argáez[2], Sigurdur Hafstein[2] and Holger Wendland[3]
[1]*Department of Mathematics, University of Sussex, Falmer, BN1 9QH, U.K.*
[2]*Science Institute and Faculty of Physical Sciences, University of Iceland, Dunhagi 5, 107 Reykjavík, Iceland*
[3]*Applied and Numerical Analysis, Department of Mathematics, University of Bayreuth, 95440 Bayreuth, Germany*

Keywords: Dynamical System, Complete Lyapunov Function, Quadratic Programming, Meshless Collocation.

Abstract: A complete Lyapunov function characterizes the behaviour of a general dynamical system. In particular, the state space is split into the chain-recurrent set, where the function is constant, and the part characterizing the gradient-like flow, where the function is strictly decreasing along solutions. Moreover, the level sets of a complete Lyapunov function provide information about the stability of connected components of the chain-recurrent set and the basin of attraction of attractors therein. In a previous method, a complete Lyapunov function was constructed by approximating the solution of the PDE $V'(\mathbf{x}) = -1$, where $'$ denotes the orbital derivative, by meshfree collocation. We propose a new method to compute a complete Lyapunov function: we only fix the orbital derivative $V'(\mathbf{x}_0) = -1$ at one point, impose the constraints $V'(\mathbf{x}) \leq 0$ for all other collocation points and minimize the corresponding reproducing kernel Hilbert space norm. We show that the problem has a unique solution which can be computed as the solution of a quadratic programming problem. The new method is applied to examples which show an improvement compared to previous methods.

1 INTRODUCTION

We will consider a general autonomous ODE

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \text{ where } \mathbf{x} \in \mathbb{R}^n. \tag{1}$$

A complete Lyapunov is a function $V\colon \mathbb{R}^n \to \mathbb{R}$ which is non-increasing along solutions of (1). The larger the area of the state space, where V is strictly decreasing, the more information can be obtained from the complete Lyapunov function.

A previous method to construct a complete Lyapunov function (Argáez et al., 2017) approximated the solution of the PDE

$$V'(\mathbf{x}) \quad = \quad -1 \tag{2}$$

using meshfree collocation. Here, $V'(\mathbf{x})$ denotes the orbital derivative, i.e. the derivative along solutions of (1); this corresponds to a function which is strictly decreasing everywhere. However, on the chain-recurrent set, including equilibria and periodic orbits, such a function does not exist. The approximation with meshfree collocation fixes finitely many collocation points $X \subset \mathbb{R}^n$, and computes the norm-minimal function such that the PDE (2) is fulfilled at these points. It computes a solution, even if the underlying PDE does not have a solution.

We propose a new approach where we split the collocation points into two sets $X = X^- \cup X^0$. We then search for the norm-minimal function v satisfying

$$v'(\mathbf{x}) \qquad \begin{cases} = -1 & \text{for } \mathbf{x} \in X^-, \\ \leq 0 & \text{for } \mathbf{x} \in X^0. \end{cases}$$

We will show in this paper that the solution of this minimization problem is unique and can be computed as the solution of a quadratic optimization problem. We choose X^- to be one point and apply the method to two examples.

Let us give an overview over the paper: in Section 2 we introduce complete Lyapunov functions and in Section 3 we review previous construction methods for complete Lyapunov functions. In Section 4 we explain the above mentioned construction method using meshfree collocation. Section 5 contains the main result of the paper, introducing our new method and the proof that the norm-minimal solution of a generalized problem with equality and inequality constraints can be computed as the solution of a quadratic programming problem. In Section 6 we apply the method to two examples before we conclude in Section 7.

Giesl, P., Argáez, C., Hafstein, S. and Wendland, H.
Construction of a Complete Lyapunov Function using Quadratic Programming.
DOI: 10.5220/0006944305600568
In *Proceedings of the 15th International Conference on Informatics in Control, Automation and Robotics (ICINCO 2018) - Volume 1*, pages 560-568
ISBN: 978-989-758-321-6

2 COMPLETE LYAPUNOV FUNCTIONS

A complete Lyapunov function for the general ODE (1) is a function $V : \mathbb{R}^n \to \mathbb{R}$ which is not increasing along solutions of (1). If V is sufficiently smooth, e.g. C^1, then this can be expressed by $V'(\mathbf{x}) \leq 0$, where $V'(\mathbf{x}) = \nabla V(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x})$ denotes the orbital derivative, i.e. the derivative along solutions of (1).

A constant function would trivially satisfy this assumption, and thus a complete Lyapunov function is required to only be constant on each connected component of the chain-recurrent set, including local attractors and repellers, and be strictly decreasing elsewhere. A point is in the chain-recurrent set, if an ε-trajectory through it comes back to it after any given time. An ε-trajectory is arbitrarily close to a true solution of the system. This indicates recurrent motion; for a precise definition see, e.g. (Conley, 1978). The dynamics outside the chain-recurrent set are similar to a gradient system, i.e. a system (1) where the right-hand side $\mathbf{f}(\mathbf{x}) = \nabla W(\mathbf{x})$ is given by the gradient of a function $W : \mathbb{R}^n \to \mathbb{R}$.

The values and level sets of the complete Lyapunov function provide additional information about the dynamics and the long-term behaviour of the system, e.g. an asymptotically stable equilibrium is a local minimum of a complete Lyapunov function. Moreover, the (constant) values of a complete Lyapunov function on each connected component of the chain-recurrent set describe the dynamics between them.

Summarizing, a smooth complete Lyapunov function satisfies

$$V'(\mathbf{x}) < 0 \text{ for } \mathbf{x} \in G, \quad (3)$$
$$V'(\mathbf{x}) = 0 \text{ for } \mathbf{x} \in C, \quad (4)$$

where G denotes the gradient-flow like set and C the chain recurrent set.

The first proof of the existence of a complete Lyapunov function for dynamical systems was given by Conley (Conley, 1978). This proof holds for a compact metric space. It considers each corresponding attractor-repeller pair and constructs a function which is 1 on the repeller, 0 on the attractor and decreasing in between. Then these functions are summed up over all attractor-repeller pairs. Later, Hurley generalized these ideas to more general spaces (Hurley, 1992; Hurley, 1998). These functions, however, are just continuous functions.

3 PREVIOUS CONSTRUCTION METHODS

Computational approaches to construct complete Lyapunov functions were proposed in (Kalies et al., 2005; Ban and Kalies, 2006; Goullet et al., 2015). The state space was subdivided into cells, defining a discrete-time system given by the multivalued time-T map between them, which was computed using the computer package GAIO (Dellnitz et al., 2001). Hence, an approximate complete Lyapunov function was constructed using graph algorithms (Ban and Kalies, 2006). This approach requires a high number of cells even for low dimensions.

In (Björnsson et al., 2015), a complete Lyapunov function was constructed as a continuous piecewise affine (CPA) function, affine on a fixed simplicial complex. However, it is assumed that information about local attractors is available, while we do not require any information.

In (Argáez et al., 2017; Argáez et al., 2018a; Argáez et al., 2018b) a complete Lyapunov function was computed by approximately solving the PDE

$$V'(\mathbf{x}) = -1 \quad (5)$$

using meshfree collocation, in particular using Radial Basis Functions; for a detailed description of the method see Section 4. This is inspired by constructing classical Lyapunov functions for an equilibrium (Giesl, 2007; Giesl and Wendland, 2007). However, (5) cannot be fulfilled at all $\mathbf{x} \in C$. Meshfree collocation still constructs an approximation, but error estimates are not available, as they compare the approximation to the solution of the problem, which does not exist. In practice, the method works well on the gradient-like part and is able to detect the chain-recurrent set as the area of the state space where the approximation fails.

Collocation points where the approximation fails are detected by comparing the orbital derivative of the approximation v to the prescribed value -1 in test points $\mathbf{y}$ near the collocation point $\mathbf{x}_j$. A collocation point $\mathbf{x}_j$ is classified as failing if $v'(\mathbf{y}) \geq -\gamma$ holds for at least one of the test points $\mathbf{y}$ near $\mathbf{x}_j$, where $0 > -\gamma > -1$ is a threshold parameter. Thus, the set of collocation points X is separated into $X = X^- \cup X^0$, where X^0 denotes the failing points and X^- the remaining ones. X^0 is an approximation of the chain-recurrent set, while X^- approximates the gradient-like set. In a subsequent step, the problem

$$V'(\mathbf{x}) = \begin{cases} -1 & \text{for } \mathbf{x} \in X^- \\ 0 & \text{for } \mathbf{x} \in X^0 \end{cases} \quad (6)$$

is solved using meshfree collocation. This step is iterated until no points are moved from X^- to X^0. Further improvements of the method deal with the fact that the right-hand side of (6) is discontinuous. Moreover, rescaling of the right-hand side $\mathbf{f}$ in (1) also improved the detection of the chain-recurrent set.

4 MESHFREE COLLOCATION

Meshfree collocation, in particular with Radial Basis Functions, is used to solve generalized interpolation problems. A classical interpolation problem is, given finitely many points $\mathbf{x}_1,\ldots,\mathbf{x}_N \in \mathbb{R}^n$ and corresponding values $r_1,\ldots,r_N \in \mathbb{R}$, to find a function v satisfying $v(\mathbf{x}_j) = r_j$ for all $j = 1,\ldots,N$.

Solving a PDE of the form $LV(\mathbf{x}) = r(\mathbf{x})$, where L denotes a differential operator, is a generalized interpolation problem as we look for a function v satisfying $Lv(\mathbf{x}_j) = r_j$ for all $j = 1,\ldots,N$, where again finitely many points $\mathbf{x}_1,\ldots,\mathbf{x}_N \in \mathbb{R}^n$ and values $r_1 = r(\mathbf{x}_1),\ldots,r_N = r(\mathbf{x}_N) \in \mathbb{R}$ are given.

The approximating functions will belong to a Hilbert space H, which we assume to have a reproducing kernel $\Phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, given by a Radial Basis Function ψ_0 through $\Phi(\mathbf{x},\mathbf{y}) := \psi_0(\|\mathbf{x}\|_2)$.

In general, we seek to reconstruct the target function $V \in H$ from the information $r_1,\ldots,r_N \in \mathbb{R}$ generated by N linearly independent functionals $\lambda_j \in H^*$, i.e. $\lambda_j(V) = r_j$ holds for $j = 1,\ldots,N$. The optimal (norm-minimal) reconstruction of the function V is the solution of the problem

$$\min\{\|v\|_H : \lambda_j(v) = r_j, 1 \le j \le N\}.$$

It is well known (Wendland, 2005) that the optimal reconstruction can be represented as a linear combination of the Riesz representers $v_j \in H$ of the functionals and that these are given by $v_j = \lambda_j^{\mathbf{y}}\Phi(\cdot,\mathbf{y})$, i.e. the functional applied to one of the arguments of the reproducing kernel. Hence, the solution can be written as

$$v(\mathbf{x}) = \sum_{j=1}^{N} \beta_j \lambda_j^{\mathbf{y}} \Phi(\mathbf{x},\mathbf{y}), \qquad (7)$$

where the coefficients β_j are determined by the interpolation conditions $\lambda_j(v) = r_j$, $1 \le j \le N$.

Consider the PDE $V'(\mathbf{x}) = r(\mathbf{x})$, where $r(\mathbf{x})$ is a given function. We choose N points $\mathbf{x}_1,\ldots,\mathbf{x}_N \in \mathbb{R}^n$ of the state space and define functionals $\lambda_j(v) := (\delta_{\mathbf{x}_j} \circ L)^{\mathbf{x}} v = v'(\mathbf{x}_j) = \nabla v(\mathbf{x}_j) \cdot \mathbf{f}(\mathbf{x}_j)$, where L denotes the linear operator of the orbital derivative $LV(\mathbf{x}) = V'(\mathbf{x})$ and δ is Dirac's delta distribution. The information is given by the right-hand side $r_j = r(\mathbf{x}_j)$ for all $1 \le j \le N$. The approximation is then

$$v(\mathbf{x}) = \sum_{j=1}^{N} \beta_j (\delta_{\mathbf{x}_j} \circ L)^{\mathbf{y}} \Phi(\mathbf{x},\mathbf{y}), \qquad (8)$$

where the coefficients $\beta_j \in \mathbb{R}$ can be calculated by solving the system $A\beta = \mathbf{r}$ of N linear equations. Here, $r_j = r(\mathbf{x}_j)$ and A is the $N \times N$ matrix with entries

$$\begin{aligned} a_{ij} &= (\delta_{\mathbf{x}_i} \circ L)^{\mathbf{x}} (\delta_{\mathbf{x}_j} \circ L)^{\mathbf{y}} \Phi(\mathbf{x},\mathbf{y}) \\ &= \langle \lambda_i^{\mathbf{y}} \Phi(\cdot,\mathbf{y}), \lambda_j^{\mathbf{z}} \Phi(\cdot,\mathbf{z}) \rangle_H. \end{aligned} \qquad (9)$$

The matrix A is positive definite, since the $\lambda_i \in H^*$ were assumed to be linearly independent.

If the PDE has a solution V, then the error $\|LV - Lv\|_{L_\infty}$ can be estimated in terms of the so-called fill distance which measures how dense the points $\mathbf{x}_1,\ldots,\mathbf{x}_N$ are. For the construction of a classical Lyapunov function for an equilibrium such error estimates were derived in (Giesl, 2007; Giesl and Wendland, 2007), see also (Narcowich et al., 2005; Wendland, 2005).

The advantage of meshfree collocation over other methods for solving PDEs is that scattered points can be added, no triangulation of the state space is necessary, the approximating function is smooth and the method works in any dimension.

In this paper, we use Wendland functions (Wendland, 1995; Wendland, 1998) as Radial Basis Functions through $\psi_0(r) := \phi_{l,k}(r)$, where $k \in \mathbb{N}$ is a smoothness parameter and $l = \lfloor \frac{n}{2} \rfloor + k + 1$. Wendland functions are positive definite functions with compact support, which are polynomials on their support; the corresponding reproducing kernel Hilbert space is norm-equivalent to the Sobolev space $W_2^{k+(n+1)/2}(\mathbb{R}^n)$. They are defined by recursion: for $l \in \mathbb{N}$, $k \in \mathbb{N}_0$ we define

$$\begin{aligned} \phi_{l,0}(r) &= (1-r)_+^l \\ \phi_{l,k+1}(r) &= \int_r^1 t\phi_{l,k}(t)\,dt \end{aligned} \qquad (10)$$

for $r \in \mathbb{R}_0^+$, where $x_+ = x$ for $x \ge 0$ and $x_+ = 0$ for $x < 0$.

We define recursively $\psi_i(r) = \frac{1}{r}\frac{d\psi_{i-1}}{dr}(r)$ for $i = 1,2$ and $r > 0$. Note that $\lim_{r\to 0}\psi_i(r)$ exists if the smoothness parameter k of the Wendland function is sufficiently large. The explicit formulas for v and its orbital derivative are then, see (8)

$$v(\mathbf{x}) = \sum_{j=1}^{N} \beta_j \langle \mathbf{x}_j - \mathbf{x}, \mathbf{f}(\mathbf{x}_j) \rangle \psi_1(\|\mathbf{x}-\mathbf{x}_j\|_2),$$

$$\begin{aligned} v'(\mathbf{x}) = \sum_{j=1}^{N} \beta_j \Big[&-\psi_1(\|\mathbf{x}-\mathbf{x}_j\|_2)\langle \mathbf{f}(\mathbf{x}), \mathbf{f}(\mathbf{x}_j)\rangle \\ &+ \psi_2(\|\mathbf{x}-\mathbf{x}_j\|_2)\langle \mathbf{x}-\mathbf{x}_j, \mathbf{f}(\mathbf{x})\rangle \cdot \langle \mathbf{x}_j - \mathbf{x}, \mathbf{f}(\mathbf{x}_j)\rangle \Big] \end{aligned}$$

where $\langle \cdot,\cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^n$.

The matrix elements of A are

$$
\begin{aligned}
a_{ij} &= \psi_2(\|\mathbf{x}_i-\mathbf{x}_j\|_2)\langle \mathbf{x}_i-\mathbf{x}_j,\mathbf{f}(\mathbf{x}_i)\rangle\langle \mathbf{x}_j-\mathbf{x}_i,\mathbf{f}(\mathbf{x}_j)\rangle \\
&\quad -\psi_1(\|\mathbf{x}_i-\mathbf{x}_j\|_2)\langle \mathbf{f}(\mathbf{x}_i),\mathbf{f}(\mathbf{x}_j)\rangle \text{ for } i\neq j \quad (11)\\
a_{ii} &= -\psi_1(0)\|\mathbf{f}(\mathbf{x}_i)\|_2^2. \quad (12)
\end{aligned}
$$

More detailed explanations on this construction are given in (Giesl, 2007, Chapter 3).

Note that by (9) the function of the form $v(\mathbf{x}) = \sum_{j=1}^N \beta_j\lambda_j^{\mathbf{y}}\Phi(\mathbf{x},\mathbf{y})$ has the following norm

$$
\begin{aligned}
\|v\|_H^2 &= \left\langle \sum_{i=1}^N \beta_i\lambda_i^{\mathbf{y}}\Phi(\cdot,\mathbf{y}), \sum_{j=1}^N \beta_j\lambda_j^{\mathbf{z}}\Phi(\cdot,\mathbf{z})\right\rangle_H \\
&= \sum_{i,j=1}^N \beta_i\beta_j\langle\lambda_i^{\mathbf{y}}\Phi(\cdot,\mathbf{y}),\lambda_j^{\mathbf{z}}\Phi(\cdot,\mathbf{z})\rangle_H \\
&= \sum_{i,j=1}^N \beta_i\beta_j a_{ij} \\
&= \beta^T A\beta. \quad (13)
\end{aligned}
$$

If the collocation points are pairwise distinct and no collocation point $\mathbf{x}_j$ is an equilibrium for the system, i.e. $\mathbf{f}(\mathbf{x}_j)\neq\mathbf{0}$ for all j, then the λ_j are linearly independent and the matrix A is positive definite; hence, the equation $A\beta=\mathbf{r}$ has a unique solution. Note that this holds true independent of whether the underlying discretized PDE has a solution or not, while the error estimates are only available if the PDE has a solution.

5 CONSTRUCTION VIA QUADRATIC PROGRAMMING

In this section we propose a new method to construct a complete Lyapunov function. The previous method, using meshfree collocation to solve the PDE (5) had the disadvantage that the PDE does not have a solution, and thus error estimates are not available. Even when assuming that we approximate the chain-recurrent set well in a first step by splitting the collocation points into failing points X^0 and the other points X^-, we then have the problem of smoothing the right-hand side of (6).

In our new approach we consider the definition of a complete Lyapunov function, using inequalities rather than equalities. In particular, a complete Lyapunov function V needs to satisfy $V'(\mathbf{x})\leq 0$, i.e. V is non-increasing. We replace (6) by the requirement

$$
V'(\mathbf{x}) \quad \begin{cases} =-1 & \text{for } \mathbf{x}\in X^- \\ \leq 0 & \text{for } \mathbf{x}\in X^0 \end{cases} \quad (14)
$$

which allows for a smooth right-hand side.

Since we have replaced the equality with inequality constraints, we can minimize an objective function. Since we seek to generalize the meshfree collocation, we will minimize the norm of V in H. For given sets of points

$$
\begin{aligned}
X^- &= \{\mathbf{x}_{-M+1},\ldots,\mathbf{x}_0\} \quad (15)\\
X^0 &= \{\mathbf{x}_1,\ldots,\mathbf{x}_N\} \quad (16)
\end{aligned}
$$

we will later set $\lambda_j(v)=v'(\mathbf{x}_j)$ and show that the norm-minimal solution of (14) exists and is the solution of a quadratic programming problem.

In (Schaback and Werner, 2006), a minimization problem in the context of classical interpolation is considered, the existence and uniqueness of the minimizer is shown and its relation to a quadratic programming problem is studied.

We, however, will consider the more general problem

$$
\begin{cases}
\text{minimize} & \|v\|_H \\
\text{subject to} & \lambda_j(v) = r_j, \\
& \qquad j=-M+1,\ldots,0 \\
& \lambda_i(v) \leq b_i,\ i=1,\ldots,N
\end{cases} \quad (17)
$$

where $r_j,b_i\in\mathbb{R}$, H is a Hilbert space with inner product $\langle\cdot,\cdot\rangle_H$ and norm $\|\cdot\|_H$, and $\lambda_i\in H^*$ are linearly independent. We first show that the problem has at most one solution.

Lemma 5.1. *The problem* (17) *has at most one solution.*

Proof. First assume that $s\in H$ is a minimizer and $v\in H$ is a solution of

$$
\begin{cases}
\lambda_j(v) = r_j,\ j=-M+1,\ldots,0 \\
\lambda_i(v) \leq b_i,\ i=1,\ldots,N.
\end{cases} \quad (18)
$$

Then $\langle s,v-s\rangle_H\geq 0$. Indeed, assume that $\langle s,v-s\rangle_H<0$. Let $\alpha\in[0,1]$. Note that $t=\alpha v+(1-\alpha)s$ satisfies (18) and we have

$$
\begin{aligned}
\|t\|_H^2 &= \|s+\alpha(v-s)\|_H^2 \\
&= \|s\|_H^2+2\alpha\langle s,v-s\rangle_H+\alpha^2\|v-s\|_H^2 \\
&< \|s\|_H^2
\end{aligned}
$$

for a suitable $\alpha>0$. This is a contradiction to s being a minimizer.

Now let $s_1,s_2\in H$ be minimizers. Then by the above argument we have $\langle s_1,s_2-s_1\rangle_H\geq 0$ and $\langle s_2,s_1-s_2\rangle_H\geq 0$. This implies

$$
\begin{aligned}
0 &\leq \|s_1-s_2\|_H^2 \\
&= \langle s_1,s_1-s_2\rangle_H-\langle s_2,s_1-s_2\rangle_H \\
&\leq 0
\end{aligned}
$$

which shows $s_1=s_2$. □

Now let H be a reproducing kernel Hilbert space with kernel Φ. We wish to show that the solution s^* of (17) is of the form

$$s^*(\mathbf{x}) \quad = \quad \sum_{j=1}^{N} \beta_j \lambda_j^{\mathbf{y}} \Phi(\mathbf{x},\mathbf{y}), \tag{19}$$

where the coefficients β_j satisfy

$$\left\{ \begin{array}{lll} \text{minimize} & \beta^T A \beta & \\ \text{subject to} & B_- \beta = \mathbf{r}_1 & \in \mathbb{R}^M \\ \text{and} & B_0 \beta \leq \mathbf{b} & \in \mathbb{R}^N. \end{array} \right. \tag{20}$$

Here, the inequality is to be read componentwise, $(\mathbf{r}_1)_j = r_j$ for $j = -M+1,\ldots,0$, $\mathbf{b} = (b_j)_{j=1,\ldots,N}$, the matrix elements a_{ij} are defined by

$$a_{ij} = \lambda_i^{\mathbf{x}} \lambda_j^{\mathbf{y}} \Phi(\mathbf{x},\mathbf{y})$$

and the matrices by

$$\begin{aligned} A &= (a_{ij})_{i,j=-M+1,\ldots,N} \in \mathbb{R}^{(M+N)\times(M+N)} \\ &= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \\ B_- &= (a_{ij})_{i=-M+1,\ldots,0,\, j=-M+1,\ldots,N} \in \mathbb{R}^{M\times(M+N)} \\ &= \begin{pmatrix} A_{11} & A_{12} \end{pmatrix} \\ B_0 &= (a_{ij})_{i=1,\ldots,N,\, j=-M+1,\ldots,N} \in \mathbb{R}^{N\times(M+N)} \\ &= \begin{pmatrix} A_{21} & A_{22} \end{pmatrix}. \end{aligned}$$

Since the λ_i are linearly independent, the matrices A, A_{11} and A_{22} are symmetric and positive definite, i.e. in particular $A_{11}^T = A_{11}$ and $A_{12}^T = A_{21}$, since they are part of the symmetric matrix A. A symmetric matrix A is positive definite if and only if $P^T A P$ is positive definite for any non-singular matrix P. Indeed, $\mathbf{v}^T A \mathbf{v} > 0$ for all $\mathbf{v} \neq \mathbf{0}$ is equivalent, using $\mathbf{v} = P\mathbf{w}$, to $\mathbf{w}^T P^T A P \mathbf{w} > 0$ for all $\mathbf{w} \neq \mathbf{0}$.

Using $P = \begin{pmatrix} I & -A_{11}^{-1} A_{12} \\ 0 & I \end{pmatrix}$ we thus have that

$$\begin{pmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} I & -A_{11}^{-1}A_{12} \\ 0 & I \end{pmatrix} = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{pmatrix}$$

is positive definite. In particular $A_{22} - A_{21}A_{11}^{-1}A_{12}$ and its inverse $Q := (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}$ are positive definite.

Let us show that the problem (20) has a unique solution.

Lemma 5.2. *Problem* (20) *has a unique solution.*

Proof. Denote $\alpha_1 = (\beta_{-M+1},\ldots,\beta_0)$ and $\alpha_2 = (\beta_1,\ldots,\beta_N)$ and introduce the new variables $\mathbf{r}_2 \in \mathbb{R}^N$. We rewrite (20) as

$$\left\{ \begin{array}{lll} \text{minimize} & \mathbf{r}_1^T \alpha_1 + \mathbf{r}_2^T \alpha_2 & \\ \text{subject to} & A_{11}\alpha_1 + A_{12}\alpha_2 & = \mathbf{r}_1 \\ & A_{21}\alpha_1 + A_{22}\alpha_2 & = \mathbf{r}_2 \\ & \mathbf{r}_2 & \leq \mathbf{b}. \end{array} \right. \tag{21}$$

We solve the equality constraints and obtain

$$\begin{aligned} \alpha_1 &= A_{11}^{-1}\mathbf{r}_1 - A_{11}^{-1}A_{12}[A_{22} - A_{21}A_{11}^{-1}A_{12}]^{-1} \\ &\quad (\mathbf{r}_2 - A_{21}A_{11}^{-1}\mathbf{r}_1), \\ \alpha_2 &= [A_{22} - A_{21}A_{11}^{-1}A_{12}]^{-1}(\mathbf{r}_2 - A_{21}A_{11}^{-1}\mathbf{r}_1). \end{aligned}$$

Thus, (21) is equivalent to

$$\left\{ \begin{array}{ll} \text{minimize} & h(\mathbf{r}_2) \\ \text{subject to} & \mathbf{r}_2 \leq \mathbf{b}, \end{array} \right. \tag{22}$$

where

$$\begin{aligned} h(\mathbf{r}_2) &= \mathbf{r}_1^T A_{11}^{-1}\mathbf{r}_1 - \mathbf{r}_1^T A_{11}^{-1} A_{12} Q \\ &\quad (\mathbf{r}_2 - A_{21}A_{11}^{-1}\mathbf{r}_1) \\ &\quad + \mathbf{r}_2^T Q(\mathbf{r}_2 - A_{21}A_{11}^{-1}\mathbf{r}_1) \\ &= \mathbf{r}_2^T Q \mathbf{r}_2 + \mathbf{v}^T \mathbf{r}_2 + c, \\ \mathbf{v} &= -2QA_{21}A_{11}^{-1}\mathbf{r}_1, \\ c &= \mathbf{r}_1^T [A_{11}^{-1} + A_{11}^{-1}A_{12}QA_{21}A_{11}^{-1}]\mathbf{r}_1 \end{aligned}$$

and Q was defined earlier. Since h is a quadratic form with positive definite matrix Q, it is strictly convex and thus the problem (22) has a unique solution, if it is feasible, which is trivially clear by choosing $\mathbf{r}_2 = \mathbf{b}$. □

Now we will show that the minimizer of (17) is of the form s^*, see (19), where the coefficients β are uniquely defined by the solution of the minimization problem (20).

Lemma 5.3. *The minimizer of problem of* (17) *is of the form* (19), *where the coefficients* β *are uniquely defined by the solution of the minimization problem* (20).

Proof. Define s^* by (19), where the coefficients β are uniquely defined by the solution of the minimization problem (20), see Lemma 5.2. Let $s \in H$ be any function satisfying the constraints (18). We seek to show that $\|s^*\|_H \leq \|s\|_H$.

The function s satisfies

$$\left\{ \begin{array}{ll} \lambda_j(s) & = (\mathbf{r}_1)_j,\ j = -M+1,\ldots,0 \\ \lambda_i(s) & = (\mathbf{r}_2)_i \leq b_i,\ i = 1,\ldots,N \end{array} \right.$$

with certain values $\mathbf{r}_2 \in \mathbb{R}^N$.

Consider now the generalized interpolation problem with the same $\mathbf{r}_1, \mathbf{r}_2$

$$\left\{ \begin{array}{ll} \text{minimize} & \|v\|_H \\ \text{subject to} & \lambda_j(v) = (\mathbf{r}_1)_j,\ j = -M+1,\ldots,0 \\ & \lambda_i(v) = (\mathbf{r}_2)_i,\ i = 1,\ldots,N. \end{array} \right.$$

By classical arguments, see (Wendland, 2005), the minimizer of this problem is given by

$$\tilde{s}(\mathbf{x}) = \sum_{j=1}^{N} \tilde{\beta}_j \lambda_j^{\mathbf{y}} \Phi(\mathbf{x}, \mathbf{y})$$

where $A\tilde{\beta} = \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{pmatrix}$. This shows that $\|\tilde{s}\|_H \leq \|s\|_H$.

Now both s^* and $\tilde{s}$ are of the form (19) and the coefficients β, $\tilde{\beta}$ both satisfy the constraints of problem (20), namely

$$\begin{cases} & B_-\beta & = \mathbf{r}_1 & \in \mathbb{R}^M \\ \text{and} & B_0\beta & \leq \mathbf{b} & \in \mathbb{R}^N. \end{cases}$$

However, the coefficients β of s^* minimize $\beta^T A \beta = \|s^*\|_H^2$ so that $\|s^*\|_H^2 \leq \|s\|_H^2 = \tilde{\beta}^T A \tilde{\beta}$, see (13). This altogether shows that

$$\|s^*\|_H \leq \|\tilde{s}\|_H \leq \|s\|_H.$$

Since the minimizer of (17) is unique by Lemma 5.1, s^* is that minimizer. □

For our application to construct complete Lyapunov functions we choose $\mathbf{r}_1 = -\mathbf{1}$ and $\mathbf{b} = \mathbf{0}$, see (14), as well as $\lambda_i = \delta_{\mathbf{x}_i} \circ L$, $i = -M+1, \ldots, N$, where L denotes operator of the orbital derivative and the points are defined in (15) and (16). Then (20) defines a quadratic programming problem, the solution of which constructs a complete Lyapunov function. The formulas for the matrix entries a_{ij} are given by (11) and (12).

Any choice of the collocation points X and the distribution between X^- and X^0 is possible, as long as they are all pairwise distinct and no equilibria. The choice $X^0 = \varnothing$ (no inequality constraint) leads to (5), while $X^- = \varnothing$ (no equality constraint) would give the trivial solution $V(\mathbf{x}) = 0$.

In this paper we choose $X^- = \{\mathbf{x}_0\}$ to be one point ($M = 1$) and N further, pairwise distinct points in the set X^0. This assumes the least amount of information which still results in a nontrivial result. Moreover, since the existence proof for a complete Lyapunov function does not allow us to fix the orbital derivative to -1 in the entire gradient-like flow part and, moreover, we do not know where the chain-recurrent set is, a solution of the problem with more points in X^- will in general not exist. However, assuming that the point $\mathbf{x}_0$ lies in the gradient-like part, we expect to be able to prove the existence of a complete Lyapunov function with one equality and many inequality constraints in future work.

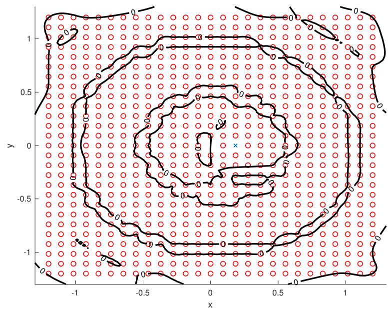

Figure 1: Points in X: only one point $(0.1846, 0)$ (marked x) in the set X^- with equality constraint $v'(x,y) = -1$ while the other points (marked ∘) in the set X^0 satisfy inequality constraints $v'(x,y) \leq 0$. Moreover, the level set $v'(x,y) = 0$ is shown, approximating the chain-recurrent set.

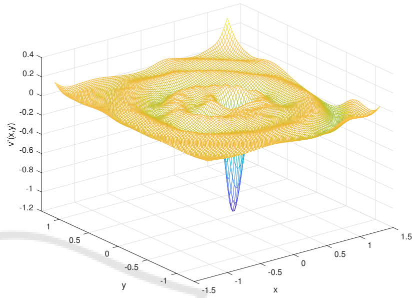

Figure 2: The orbital derivative $v'(x,y)$ of the constructed complete Lyapunov function v. v' is approximately zero on the chain-recurrent set (origin and the two periodic orbits, spheres with radius 0.5 and 1, respectively) and negative everywhere else. Note that $v'(0.1846, 0) = -1$ by the equality constraint.

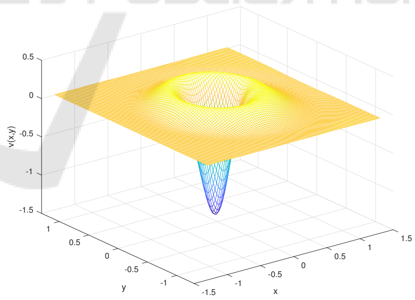

Figure 3: The constructed complete Lyapunov function $v(x,y)$. v has a minimum at the asymptotically stable equilibrium at the origin, a maximum at the unstable periodic orbit (sphere of radius 0.5) and a minimum at the asymptotically stable periodic orbit (sphere of radius 1).

6 EXAMPLES

6.1 Two Periodic Orbits

We consider the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -x(x^2+y^2-1/4)(x^2+y^2-1) - y \\ -y(x^2+y^2-1/4)(x^2+y^2-1) + x \end{pmatrix}. \quad (23)$$

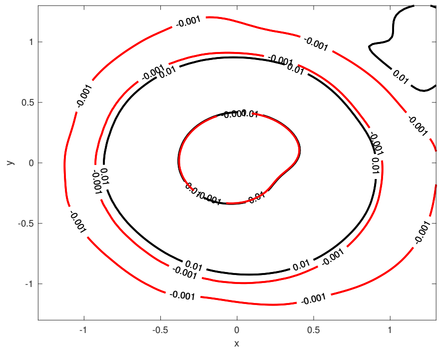

Figure 4: Level sets of the constructed complete Lyapunov function $v(x,y)$ with level 0.01 (black, maximum) and -0.001 (red, minimum). This shows that v has a maximum at the unstable periodic orbit of radius 0.5 and a minimum at the asymptotically stable periodic orbit of radius 1.

This system has an asymptotically stable equilibrium at the origin as well as two periodic orbits: an asymptotically stable periodic orbit at $\Omega_1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 1\}$ and a repelling periodic orbit at $\Omega_2 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = (0.5)^2\}$.

For the quadratic programming we have used the points $X = \left(\frac{1.2}{13}\mathbb{Z}^2 \cap [-1.2,1.2]\right) \setminus \{(0,0)\}$ consisting of 728 points, excluding the equilibrium, and split them into $X^- = \{(0.1846,0)\}$ ($M = 1$ point) and $X^0 = X \setminus X^-$ ($N = 727$ points). We have used the Wendland function $\psi_0(r) = \phi_{6,4}(r) = (1-r)_+^6(35r^2+18r+3)$.

Figure 1 shows the points marked with a circle (X^0) and a cross (X^-) as well as the level set of $v'(x,y) = 0$, where v is the constructed complete Lyapunov function. Figure 2 displays the orbital derivative v' which is -1 at the point $(0.1846,0)$ and approximately zero on the equilibrium and the two periodic orbits (spheres of radius 0.5 and 1). Figure 3 shows the constructed complete Lyapunov function with a minimum at the origin, which is an asymptotically stable equilibrium, a maximum at the unstable periodic orbit (sphere of radius 0.5) and a minimum at the attracting periodic orbit (sphere of radius 1). The latter minimum can be better seen in Figure 4, where some level sets of v are displayed.

Note that it is surprising to detect the periodic orbit with radius 1, although the only point where we set the orbital derivative to be -1 is inside circle with radius 0.5, bounded by the unstable periodic orbit. A complete Lyapunov function could have been constant on $\{(x,y) \in \mathbb{R}^2 \mid \sqrt{x^2+y^2} \geq 0.5\}$, but this is not the minimizing solution.

Compared to previous constructions of complete Lyapunov functions for this example, see (Argáez et al., 2017, §4.1) and (Argáez et al., 2018a, §2.1), the new method manages to construct a complete Lyapunov function using only 728 collocation points that compares favorably with a complete Lyapunov function that is constructed by solving a system of linear equations using 29,440 collocation points.

6.2 Homoclinic Orbit

We consider the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, where $\mathbf{f}(x,y)$ is given by

$$\begin{pmatrix} x(1-x^2-y^2) - y((x-1)^2 + (x^2+y^2-1)^2) \\ y(1-x^2-y^2) + x((x-1)^2 + (x^2+y^2-1)^2) \end{pmatrix}. \tag{24}$$

The origin is an unstable focus and the system has an asymptotically stable homoclinic orbit at a circle centered at the origin and with radius 1, connecting the equilibrium $(1,0)$ with itself.

For the optimization we have used the points $X = \left(\frac{1.2}{13}\mathbb{Z}^2 \cap [-1.2,1.2]\right) \setminus \{(0,0)\}$ consisting of 728 points, excluding the two equilibria and split them into $X^- = \{(0.1846,0)\}$ ($M = 1$ point) and $X^0 = X \setminus X^-$ ($N = 727$ points). We have used the Wendland function $\psi_0(r) = \phi_{6,4}(r) = (1-r)_+^6(35r^2+18r+3)$.

Figure 5 shows the points marked with a circle (X^0) and a cross (X^-) as well as the level set of $v'(x,y) = -0.0001$, where v is the constructed complete Lyapunov function. Figure 6 displays the orbital derivative v' which is -1 at the point $(0.1846,0)$ and approximately zero on the homoclinic orbit and the equilibrium at the origin. Figure 7 shows the constructed complete Lyapunov function with a maximum at the origin and a minimum at the homoclinic orbit, which is attractive.

Again our novel method compares favorably to solving a system linear equations using almost 20 times more collocation points, see (Argáez et al., 2017, §4.3) and (Argáez et al., 2018a, §2.3), where a complete Lyapunov function that is constructed by solving a system linear equations using 29,440 collocation points.

7 CONCLUSIONS

In this paper we have introduced a new method to construct complete Lyapunov functions. Complete Lyapunov functions characterize the complete dynamics of a dynamical system by separating the state space into the part of the gradient-like flow and the chain-recurrent set. A complete Lyapunov function has a strictly negative orbital derivative in the gradient-like part, while the orbital derivative is zero in the chain-recurrent set.

For the construction method, one point in the gradient-like set is chosen, where the orbital deriva-

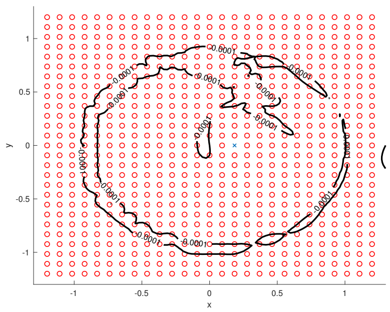

Figure 5: Points in X: only one point $(0.1846,0)$ (marked x) in the set X^- with equality constraint $v'(x,y)=-1$ while the other points (marked ∘) in the set X^0 satisfy inequality constraints $v'(x,y)\leq 0$. Moreover, the level set $v'(x,y)=-0.0001$ is shown, approximating the chain-recurrent set.

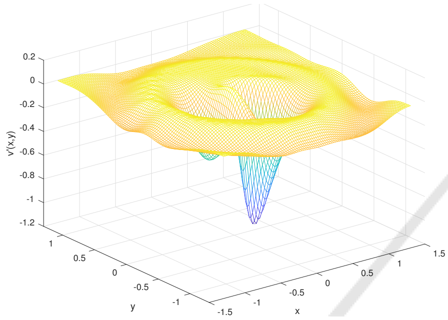

Figure 6: The orbital derivative $v'(x,y)$ of the constructed complete Lyapunov function v. v' is approximately zero on the chain-recurrent set (origin and homoclinic orbit) and negative everywhere else. Note that $v'(0.1846,0)=-1$ by the equality constraint.

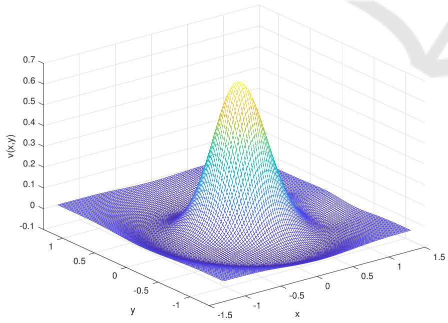

Figure 7: The constructed complete Lyapunov function $v(x,y)$. v has a maximum at the unstable equilibrium at the origin and a minimum at the homoclinic orbit at the unit sphere.

tive is fixed to be -1, and further points are fixed, where the orbital derivative is bounded from above by 0. We have formulated this as a minimizing problem, searching for a function in a reproducing kernel Hilbert space, which satisfies one equality and finitely many inequality constraints. Furthermore, we have shown that this minimization problem in a reproducing kernel Hilbert space has a unique solution and that it is of a specific form which can be computed as quadratic programming problem. The proposed method was able to successfully construct complete Lyapunov functions for two test examples with few points.

Possible extensions include different distributions of the points in X between X^- and X^0: while previous methods had all points in X^-, in this paper X^- consists of only one point. In future work we will develop a methodology to split the points between these two sets and combine it with an iterative method, moving points between these two sets.

ACKNOWLEDGEMENTS

The research in this paper is supported by the Icelandic Research Fund (Rannís) grant number 163074-052, Complete Lyapunov functions: Efficient numerical computation.

REFERENCES

Argáez, C., Giesl, P., and Hafstein, S. (2017). Analysing dynamical systems towards computing complete Lyapunov functions. In *Proceedings of the 7th International Conference on Simulation and Modeling Methodologies, Technologies and Applications, Madrid, Spain*, pages 323–330.

Argáez, C., Giesl, P., and Hafstein, S. (2018a). Computational approach for complete Lyapunov functions. accepted for publication in Springer Proceedings in Mathematics and Statistics Series 2018.

Argáez, C., Giesl, P., and Hafstein, S. (2018b). Iterative construction of complete Lyapunov functions. accepted for publication in Proceedings of the 8th International Conference on Simulation and Modeling Methodologies, Technologies and Applications, Porto, Portugal.

Ban, H. and Kalies, W. (2006). A computational approach to Conley's decomposition theorem. *J. Comput. Nonlinear Dynam*, 1(4):312–319.

Björnsson, J., Giesl, P., Hafstein, S., Kellett, C., and Li, H. (2015). Computation of Lyapunov functions for systems with multiple attractors. *Discrete Contin. Dyn. Syst. Ser. A*, 35(9):4019–4039.

Conley, C. (1978). *Isolated Invariant Sets and the Morse Index*. CBMS Regional Conference Series no. 38. American Mathematical Society.

Dellnitz, M., Froyland, G., and Junge, O. (2001). The algorithms behind GAIO – set oriented numerical methods for dynamical systems. In *Ergodic theory, analysis, and efficient simulation of dynamical systems*, pages 145–174, 805–807. Springer, Berlin.

Giesl, P. (2007). *Construction of Global Lyapunov Functions Using Radial Basis Functions*. Lecture Notes in Math. 1904, Springer.

Giesl, P. and Wendland, H. (2007). Meshless collocation: error estimates with application to Dynamical Systems. *SIAM J. Numer. Anal.*, 45(4):1723–1741.

Goullet, A., Harker, S., Mischaikow, K., Kalies, W., and Kasti, D. (2015). Efficient computation of Lyapunov functions for Morse decompositions. *Discrete Contin. Dyn. Syst.*, 20(8):2419–2451.

Hurley, M. (1992). Noncompact chain recurrence and attraction. *Proc. Amer. Math. Soc.*, 115:1139–1148.

Hurley, M. (1998). Lyapunov functions and attractors in arbitrary metric spaces. *Proc. Amer. Math. Soc.*, 126:245–256.

Kalies, W., Mischaikow, K., and VanderVorst, R. (2005). An algorithmic approach to chain recurrence. *Found. Comput. Math*, 5(4):409–449.

Narcowich, F. J., Ward, J. D., and Wendland, H. (2005). Sobolev bounds on functions with scattered zeros, with applications to radial basis function surface fitting. *Mathematics of Computation*, 74:743–763.

Schaback, R. and Werner, J. (2006). Linearly constrained reconstruction of functions by kernels with applications to machine learning. *Adv. Comput. Math.*, 25(1-3):237–258.

Wendland, H. (1995). Piecewise polynomial, positive definite and compactly supported radial functions of minimal degree. *Adv. Comput. Math.*, 4(4):389–396.

Wendland, H. (1998). Error estimates for interpolation by compactly supported Radial Basis Functions of minimal degree. *J. Approx. Theory*, 93:258–272.

Wendland, H. (2005). *Scattered data approximation*, volume 17 of *Cambridge Monographs on Applied and Computational Mathematics*. Cambridge University Press, Cambridge.